On Noether symmetries and form invariance of mechanico-electrical systems

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Abstract

This Letter focuses on form invariance and Noether symmetries of mechanico-electrical systems. Based on the invariance of Hamiltonian actions for mechanico-electrical systems under the infinitesimal transformation of the coordinates, the electric quantities and the time, the authors present the Noether symmetry transformation, the Noether quasi-symmetry transformation, the generalized Noether quasi-symmetry transformation and the general Killing equations of Lagrange mechanico-electrical systems and Lagrange–Maxwell mechanico-electrical systems. Using the invariance of the differential equations, satisfied by physical quantities, such as Lagrangian, non-potential general forces, under the infinitesimal transformation, the authors propose the definition and criterions of the form invariance for mechanico-electrical systems. The Letter also demonstrates connection between the Noether symmetries and the form invariance of mechanico-electrical systems. An example is designed to illustrate these results.

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1. Introduction

It is well known that symmetric principles are key issues in mathematics and physics. There are close relationships between symmetries and conserved quantities. Many methods have been developed to seek conserved
quantities of mechanical systems [1]. Noether first proposed the famous Noether symmetry theory which deals with the invariance of Hamiltonian actions under infinitesimal transformations [2]. Djukić and Vujanović [3] further investigated Noether symmetries and conserved quantities of non-conservative holonomic systems by including generalized velocities. Studies of Noether symmetry are very active in many areas such as mathematics, mechanics, and classical fields etc. [4–8]. For example, the Noether theory was recently applied to Birkhoffian systems, to 2D spinning particles and, along with the hyperextended Scaler-tensor theory, to the FLRW models (it is logical to consider an anisotropic and inhomogeneous model) [9–11]. Nevertheless, mechanico-electrical systems have rarely been explored in this way so far.

Recently, Mei presented a form invariance theory dealing with the invariance of the differential equation satisfied by physical quantities such as Lagrangian, non-potential generalized forces and generalized constrained forces under the infinitesimal transformation about time and generalized coordinates [12,13]. The authors studied the form invariance of the canonical and the generalized canonical Hamiltonian systems [14]. The approach is significant in seeking conserved quantities.

In this Letter, the authors generalize the Noether theory and the form invariance theory to mechanic-electrical systems, and then proposed new methods to locate conserved quantities of mechanic-electrical systems.

2. Lagrange–Maxwell equations of mechanico-electrical systems

Mechanical-electrical systems are systems in which a mechanical process and an electromagnetic process are coupled. The mechanical part, constituted by $N$ particles, is described by $n$ generalized coordinates $q_s$ ($s = 1, 2, \ldots, n$), and the electrodynmal part, constituted by $m$ electric circuits, is described by the model of electrics. Consider a mechanico-electrical system composed of $m$ circuits consisting of linear conductors and capacitors. Let $i_k$ denote the current, $u_k$ denote the electronic potential, $e_k$ ($\dot{e}_k = i_k$) denote the charge in the capacitor, $R_k$ denote the resistance, and $C_k$ denote the capacitance. Then the Lagrangian of the mechanico-electrical system is

$$L = T(q, \dot{q}) \cdot V(q) + W_m(q, e) - W_e(q, e),$$

(1)

where $T$ and $V$ are, respectively, the kinetic energy and the potential energy, and the electric field energy and the magnetic field energy of the $m$th circuit are, respectively, defined by

$$W_e = \frac{1}{2} C_k e_k^2, \quad W_m = \frac{1}{2} L_{kr} i_k i_r \quad (k, r = 1, \ldots, m),$$

(2)

where $L_{kr}$ ($k \neq r$) is the mutual inductance between the $k$th circuit and the $r$th circuit, $L_{kk}$ is self inductance of the $k$th return circuit, and repeated suffixes stand for the sum.

Motion of the system is governed by the Lagrange–Maxwell equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial F}{\partial e_k} = u_k, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial q_s} = Q_s \quad (k = 1, \ldots, m; \ s = 1, \ldots, n),$$

(3)

where $Q_s$ is generalized non-potential forces. Eq. (3) consists of $n + m$ 2-order ordinary differential equations with respect to $n$ generalized coordinates $q_s$ ($s = 1, 2, \ldots, n$) and $m$ generalized electric quantities $e_k$ ($k = 1, 2, \ldots, m$). In Eq. (3), the dissipative function $F$ of the system is

$$F = F_e(\dot{e}) + F_m(q, \dot{q}),$$

(4)

where the electric dissipative function is given by

$$F_e = \frac{1}{2} R_k i_k^2 = \frac{1}{2} R_k e_k^2 \quad (k = 1, \ldots, m)$$

(5)

and $F_m$ is the dissipative function of the viscous frictional damping force, and $Q_s$ is non-potential general force.
2.1. Variation principle of mechanico-electrical systems

The Hamiltonian action of a mechanico-electrical system is defined as

$$ S(\gamma) = \int_{t_1}^{t_2} L(t, q, \dot{q}, e, \dot{e}) \, dt, $$

where $\gamma$ is an arbitrary curve. Introduce infinitesimal transformations of the generalized coordinates, the generalized electric quantities and the time

$$ t^* = t + \Delta t, \quad q^*_i(t) = q_i(t) + \Delta q_i, \quad e^*_k = e_k + \Delta e_k \quad (s = 1, \ldots, n; \; k = 1, \ldots, m), $$

where $\Delta = \varepsilon$ is small parameter, and $\xi_\alpha^s, \xi_\alpha^e, \eta_\alpha^e$ are the infinitesimal generators. Hamiltonian action $S(\gamma^*)$ of the system under the infinitesimal transformation (8) is written as

$$ S(\gamma^*) = \int_{t_1}^{t_2} L(t^*, q^*, \dot{q}^*, e^*, \dot{e}^*) \, dt^*. $$

From Eq. (7), one obtains

$$ dt^* = dt + d(\Delta t) = \left[1 + \frac{d(\Delta t)}{dt}\right] \, dt. $$(10)

Application of the variational calculus to the Hamiltonian action $S(\gamma)$ leads to

$$ \Delta S = S(\gamma^*) - S(\gamma) \approx \int_{t_1}^{t_2} \left( \frac{d(\Delta t)}{dt} + \frac{\partial L}{\partial q_i} \Delta t + \frac{\partial L}{\partial q_i} \Delta q_i + \frac{\partial L}{\partial e_k} \Delta e_k + \frac{\partial L}{\partial e_k} \Delta e_k \right) \, dt. $$

Here only the linear term of small parameter $\varepsilon$ is taken into consideration. That is, all non-linear terms of small parameter $\varepsilon$ are neglected in the variation $\Delta S$.

The relationships between the isochronous variation and the complete variation are given by

$$ \Delta q_i = \delta q_i + \dot{q}_i \Delta t, \quad \Delta e_k = \delta e_k + \dot{e}_k \Delta t, \quad \Delta \dot{q}_i = \delta \dot{q}_i + \dot{\dot{q}}_i \Delta t, \quad \Delta \dot{e}_k = \delta \dot{e}_k + \dot{\dot{e}}_k \Delta t. $$

Substitution of Eq. (12) into (11) yields

$$ \Delta S = \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( L \Delta t + \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial e_k} \delta e_k \right) + \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \left( \frac{\partial L}{\partial e_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{e}_k} \right) \delta e_k \right\}. $$

Eq. (13) is a fundamental variation formula of the Hamiltonian action of mechanico-electrical systems. Inserting Eq. (7) into Eq. (13) leads to

$$ \Delta S = \int_{t_1}^{t_2} \varepsilon a \left\{ \frac{d}{dt} \left( L_{\xi_\alpha} + \frac{\partial L}{\partial q_i} \xi_\alpha^s + \frac{\partial L}{\partial e_k} \xi_\alpha^e \right) + \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \xi_\alpha^s + \left( \frac{\partial L}{\partial e_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{e}_k} \right) \eta_\alpha^e \right\} \, dt. $$

Eq. (14) is a variational principle of mechanical-electrical systems.
where 
\[ \tilde{\xi}^a_s = \xi^a_s - \dot{q}_s \xi^a_0, \quad \tilde{\eta}^a_k = \eta^a_k - \dot{e}_k \xi^a_0. \] (15)

3. Noether theory of the mechanico-electrical systems

It is well known that a system has the Noether symmetry when its Hamilton action is invariant under an infinitesimal transformation. The definitions and criterions of Noether symmetry transformations for mechanico-electrical systems are given as follow.

3.1. Noether symmetry transformations of mechanico-electrical systems

**Definition 1.** If the Hamiltonian action of a mechanico-electrical system is invariant under an infinitesimal transformation, and the following condition
\[ \frac{\Delta S}{\Delta t} = 0 \] (16)
always holds, then the transformation is called the Noether symmetry transformation associated with the mechanico-electrical system.

**Definition 1**, with Eqs. (13) and (14), leads to:

**Criterion 1.** Infinitesimal transformation (7) is a Noether symmetry transformation of mechanico-electrical system (3), on the condition that
\[ \frac{\partial L}{\partial t} \xi^a_0 + \frac{\partial L}{\partial q_s} \xi^a_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}^a_s + \frac{\partial L}{\partial e_k} \eta^a_k + \frac{\partial L}{\partial \dot{e}_k} \dot{\eta}^a_k + \left( L - \frac{\partial L}{\partial q_s} \dot{q}_s - \frac{\partial L}{\partial e_k} \dot{e}_k \right) \xi^a_0 = 0 \] (17)

**Criterion 2.** Infinitesimal transformation (8) is a Noether symmetry transformation of mechanico-electrical system (3), if equations
\[ \frac{d}{dt} \left( L \xi^a_0 + \frac{\partial L}{\partial q_s} \xi^a_s + \frac{\partial L}{\partial e_k} \eta^a_k + \left( L - \frac{\partial L}{\partial q_s} \dot{q}_s - \frac{\partial L}{\partial e_k} \dot{e}_k \right) \xi^a_0 \right) = 0 \] (18)
hold.

Consider the following relationships
\[ \Delta t = \varepsilon_a \xi^a_0 (t, q, q, e, \dot{e}), \quad \Delta q_s (t) = \varepsilon_a \xi^a_s (t, q, q, e, \dot{e}), \quad \Delta e_k (t) = \varepsilon_a \eta^a_k (t, q, q, e, \dot{e}), \] (19)
then Eq. (17) is written as r equations
\[ \frac{\partial L}{\partial t} \xi^a_0 + \frac{\partial L}{\partial q_s} \xi^a_s + \frac{\partial L}{\partial e_k} \eta^a_k + \left( L - \frac{\partial L}{\partial q_s} \dot{q}_s - \frac{\partial L}{\partial e_k} \dot{e}_k \right) \xi^a_0 = 0 \quad (\alpha = 1, \ldots, r). \] (20)

If let \( \alpha = 1 \) in Eq. (20), the Noether equality is obtained
\[ \frac{\partial L}{\partial t} \xi^a_0 + \frac{\partial L}{\partial q_s} \xi^a_s + \frac{\partial L}{\partial e_k} \eta^a_k + \left( L - \frac{\partial L}{\partial q_s} \dot{q}_s - \frac{\partial L}{\partial e_k} \dot{e}_k \right) \xi^a_0 = 0. \] (21)

The Noether symmetry transformation of the mechanico-electrical system can be determined by **Criterion 1** and **Criterion 2**.
3.2. Noether quasi-symmetry transformations of mechanico-electrical systems

Bessel-Hegen gave the definition of the gauge transformation of Lagrangian systems [15]. Based on the idea, the Noether quasi-symmetry transformation can be defined for mechanico-electrical systems.

For a mechanico-electrical system with its Lagrangian \( L(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}) \) and a gauge function \( G(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}) \), let

\[
L_1(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}) = L(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}) + \frac{d}{dt} G(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}). \tag{22}
\]

Then the Noether quasi-symmetry transformation of the system can be obtained through substituting Eq. (22) into Eq. (11).

**Definition 2.** If the Hamiltonian action of a mechanico-electrical system is quasi-invariant under an infinitesimal transformation (7), that is, the following condition

\[
\frac{\Delta S}{\Delta t} = - \int_{t_1}^{t_2} \frac{d}{dt} (\Delta G) \, dt \tag{23}
\]

always holds for gauge function \( G \), then the transformation is defined as Noether quasi-symmetry transformation.

According to **Definition 2**, Eqs. (13) and (14) leads to the follows propositions.

**Criterion 3.** Infinitesimal transformation (7) is the Noether quasi-symmetry transformation of a mechanico-electrical system, if the transformation satisfies the following equation

\[
\frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + \frac{\partial L}{\partial \dot{q}_s} \Delta \dot{\xi}_s + L \frac{d}{dt} (\Delta t) = - \frac{d}{dt} (\Delta G). \tag{24}
\]

**Criterion 4.** Infinitesimal transformation (8) is the Noether quasi-symmetry transformation of a mechanico-electrical system, if the \( r \) equations

\[
\frac{d}{dt} \left( L \xi_0^a + \frac{\partial L}{\partial q_s} \xi_s^a + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s^a \right) + \left( \frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \right) \xi_s^a + \left( \frac{\partial L}{\partial \dot{q}_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \right) \dot{\xi}_s^a = \frac{d G^a}{dt} \tag{25}
\]

hold.

Based on the independence of the parameter \( \varepsilon_\alpha \), Eq. (19) and the following relationship

\[
\Delta G = \varepsilon_\alpha G^\alpha, \tag{26}
\]

Eq. (24) yields \( r \) equations

\[
\frac{\partial L}{\partial t} \xi_0^a + \frac{\partial L}{\partial q_s} \xi_s^a + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s^a + \frac{\partial L}{\partial \dot{q}_s} \Delta \dot{\xi}_s^a + L \left( \frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \right) \xi_s^a = - \frac{d G^a}{dt}. \tag{27}
\]

When \( \alpha = 1 \), Eq. (27) gives the Noether equality

\[
\frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + \frac{\partial L}{\partial \dot{q}_s} \Delta \dot{\xi}_s + L \left( \frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \right) \xi_s = - \dot{G}. \tag{28}
\]

The Noether quasi-symmetry transformation of mechanico-electrical systems can be determined via **Criterion 3** and **Criterion 4** when the Lagrangian \( L \) of the system is given.
3.3. Generalized Noether quasi-symmetry transformation of the mechanico-electrical systems

Notice that \( Q_s - \partial F_s/\partial \dot{q}_s \) is the generalized force, and \( u_k - \partial F_e/\partial \dot{e}_k \) is the generalized electrokinetic potential of the return circuit. If the Lagrangian of a mechanico-electrical system satisfies

\[
\int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{e}}, \dot{\mathbf{e}}) \, dt = \int_{t_1}^{t_2} L_1(t, \mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{e}}, \dot{\mathbf{e}}) \, dt + \int_{t_1}^{t_2} \left( Q_s - \frac{\partial F}{\partial \dot{q}_s} \right) \delta q_s + \int_{t_1}^{t_2} \left( u_k - \frac{\partial F}{\partial \dot{e}_k} \right) \delta e_k,
\]

(29)
then the Hamiltonian action of the mechanico-electrical system is generalized quasi-invariant under the infinitesimal transformation (7).

**Definition 3.** An infinitesimal transformation (7) is a generalized Noether quasi-symmetry transformation, if the Hamiltonian action of a mechanico-electrical systems is generalized quasi-invariant under the infinitesimal transformation (7). That is, the following condition

\[
\Delta S = \int_{t_1}^{t_2} \left[ \frac{d}{dt} \left( \Delta L \right) + \left( Q_s - \frac{\partial F}{\partial \dot{q}_s} \right) \delta q_s + \left( u_k - \frac{\partial F}{\partial \dot{e}_k} \right) \delta e_k \right] dt
\]

holds, where \( (Q_s - \partial F/\partial \dot{q}_s) \delta q_s \) is the sum of virtual work done by generalized non-potential forces, and \( (u_k - \partial F/\partial \dot{e}_k) \delta e_k \) is the sum of virtual work done by generalized electromotive forces.

**Definition 3.** Eqs. (13) and (14) lead to the follows propositions.

**Criterion 5.** There exists a generalized Noether quasi-symmetry transformation in a mechanico-electrical system, if the infinitesimal transformation (7) satisfies the following equation

\[
\frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q_s} \Delta q_s + \frac{\partial L}{\partial \dot{q}_s} \Delta \dot{q}_s + \frac{\partial L}{\partial e_k} \Delta e_k + \frac{\partial L}{\partial \dot{e}_k} \Delta \dot{e}_k + \frac{d}{dt} \left( \Delta L \right) = 0
\]

(30)

**Criterion 6.** There exists a generalized Noether quasi-symmetry transformation in a mechanico-electrical system, if the infinitesimal transformation (8) satisfies the following equation

\[
\frac{d}{dt} \left( L \xi^a_k + \frac{\partial L}{\partial q_s} \xi^a_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}^a_s + \frac{\partial L}{\partial e_k} \eta^a_k + \frac{\partial L}{\partial \dot{e}_k} \dot{\eta}^a_k \right) = 0
\]

(32)

Due to the independence of the parameter \( \varepsilon_s \), Eqs. (19), (27) and (31) lead to

\[
\frac{\partial L}{\partial t} \xi^a_0 + \frac{\partial L}{\partial q_s} \xi^a_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}^a_s + \frac{\partial L}{\partial e_k} \eta^a_k + \frac{\partial L}{\partial \dot{e}_k} \dot{\eta}^a_k + \left( L - \frac{\partial L}{\partial q_s} \dot{q}_s - \frac{\partial L}{\partial \dot{e}_k} \dot{e}_k \right) \delta \xi^a_0 = 0
\]

(33)
The generalized Noether quasi-symmetry transformation can be determined by **Criterion 5** and **Criterion 6**. When \( \alpha = 1 \), Eq. (33) gives the generalized Noether equality
\[
\frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\eta}_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\eta}_s + \left( \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s \right) \xi_0 =
\]
\[
+ \left( Q_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\xi_s - \dot{q}_s \xi_0) + \left( u_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\eta_s - \dot{q}_s \xi_0) = -\dot{G}.
\]

(34)

3.4. Killing equations of mechanico-electrical systems

Eqs. (17), (24) and (31) can be, respectively, cast into the Killing equations or the generalized Killing equations, one-order differential equations with respect to infinitesimal generators \( \xi_0, \xi_s, \eta_s, G \).

Expanding Eq. (17), one obtains
\[
\frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\eta}_s + \left( L - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s \right) \xi_0 =
\]
\[
+ \left( Q_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\dot{\xi}_s - \dot{q}_s \xi_0) + \left( u_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\dot{\eta}_s - \dot{q}_s \xi_0) = -\dot{G}.
\]

(35)

Eqs. (35), (36) and (37) are \((n + m + 1)r\) one-order partial differential equations with respect to \((n + m + 1)r\) unknown functions \( \xi_0, \xi_s, \eta_s \). They are the Killing equations of a Lagrange mechanico-electrical system. The generators can be solved from Eqs. (35), (36) and (37), for given Lagrangian of the system.

From Eq. (24) one can analogously get
\[
\frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\eta}_s + \left( L - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s \right) \xi_0 =
\]
\[
+ \left( Q_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\dot{\xi}_s - \dot{q}_s \xi_0) + \left( u_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\dot{\eta}_s - \dot{q}_s \xi_0) = -\dot{G}.
\]

(38)

Eqs. (38), (39) and (40) are the Killing equations with a gauge function of the mechanico-electrical system, which are \((n + m + 2)r\) one-order partial differential equations with respect to \((n + m + 2)r\) unknown functions \( \xi_0, \xi_s, \eta_s \) and \( G \).

Eq. (31) yields
\[
\frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s + \frac{\partial L}{\partial \dot{q}_s} \dot{\eta}_s + \left( L - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s - \frac{\partial L}{\partial \dot{q}_s} \dot{\xi}_s \right) \xi_0 =
\]
\[
+ \left( Q_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\dot{\xi}_s - \dot{q}_s \xi_0) + \left( u_s - \frac{\partial F}{\partial \dot{q}_s} \right) (\dot{\eta}_s - \dot{q}_s \xi_0) = -\dot{G}.
\]
3.5. Noether theorems of Lagrange mechanico-electrical systems

Consider a Lagrangian mechanico-electrical system satisfying conditions \( Q_s - \partial F/\partial \dot{q}_s = 0 \) and \( u_k - \partial F/\partial \dot{e}_k = 0 \). Eq. (3) becomes

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{e}_s} - \frac{\partial L}{\partial e_s} = 0. \tag{44}
\]

**Theorem 1.** If the infinitesimal transformation (7) is a Noether quasi-symmetry transformation, then the Lagrange mechanico-electrical system possesses the conserved quantities

\[
L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_s} \eta_k + G = C \quad (s = 1, \ldots, n; \quad k = 1, \ldots, m), \tag{45}
\]

where the gauge function is determined by Eq. (28) and \( C \) is a constant.

**Proof.** A Noether quasi-symmetry transformation (7) makes Eq. (23) hold. The arbitrariness of the integral region \([t_1, t_2]\) in Eqs. (14) and (23) yields

\[
\frac{d}{dt} \left( L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_s} \eta_k + G \right) + \left( \frac{\partial L}{\partial \dot{q}_s} - \frac{d}{dt} \frac{\partial L}{\partial q_s} \right) \bar{\xi}_s + \left( \frac{\partial L}{\partial \dot{e}_s} - \frac{d}{dt} \frac{\partial L}{\partial e_s} \right) \bar{\eta}_k = 0. \tag{46}
\]

Substituting Eq. (44) into Eq. (46) leads to

\[
\frac{d}{dt} \left( L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_s} \eta_k + G \right) = 0. \tag{47}
\]

**Theorem 2.** If the infinitesimal transformation (8) is Noether symmetry transformation, the Lagrange mechanico-electrical system possesses linear independent first integral as

\[
L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_s} \eta_k = C \quad (s = 1, \ldots, n; \quad k = 1, \ldots, m). \tag{48}
\]

The proof of **Theorem 2** is similar to that of the **Theorem 1**.

**Theorem 1** and **Theorem 2** are, respectively, the generalized Noether theorem and the Noether theorem for the Lagrange mechanico-electrical system.
3.6. Noether theorems of Lagrange–Maxwell mechanico-electrical systems

For a Lagrange–Maxwell mechanico-electrical system defined by Eq. (3), the conserved quantity can be obtained from the following generalized Noether theorem.

**Theorem 3.** If the infinitesimal transformation (8) is a generalized quasi-symmetry transformation, the Lagrange–Maxwell mechanico-electrical system possesses the conserved quantities

\[
L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_k} \eta_k + G = C \quad (s = 1, \ldots, n; \ k = 1, \ldots, m),
\]

where the gauge function \( G \) is given by Eq. (34) and \( C \) is a constant.

**Proof.** The generalized quasi-symmetry transformation makes Eq. (30) hold. Due to Eq. (14), Eq. (30) can be expressed as

\[
\int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left( L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_k} \eta_k + G \right) + \left( \frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} + \frac{\partial F}{\partial q_s} \right) \xi_s + \left( \frac{\partial L}{\partial e_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{e}_k} + \frac{\partial F}{\partial e_k} \right) \eta_k \right\} dt = 0.
\]

Substituting Eq. (8) into Eq. (50), and using arbitrariness of the integral region \([t_1, t_2]\), one obtains

\[
\frac{d}{dt} \left( L\xi_0 + \frac{\partial L}{\partial \dot{q}_s} \xi_s + \frac{\partial L}{\partial \dot{e}_k} \eta_k + G \right) = 0. \quad \square
\]

Theorem 3 is the generalized Noether theorem of the Lagrange–Maxwell mechanico-electrical systems. Theorem 1 and Theorem 2 of the Lagrange mechanico-electrical systems are special cases of the Theorem 3.

4. Definitions and criterions of form invariance for mechanico-electrical systems

Introduce a point transformation with respect to the general coordinates and the general electric quantities and time

\[
\tau^* = \tau + \varepsilon \xi_0(t, q, \dot{q}, \dot{e}, \ddot{e}), \quad q^*_s = q_s + \varepsilon \xi_s(t, q, \dot{q}, \dot{e}, \ddot{e}), \quad e^*_k = e_k + \varepsilon \eta_k(t, q, \dot{q}, \dot{e}, \ddot{e}),
\]

where \( \varepsilon \) is a small parameter, \( \xi_0, \xi_s, \eta_k \) are infinitesimal generators.

Under the infinitesimal transformation (52), the physical quantities such as the Lagrangian \( L = L(t, q, \dot{q}, \dot{e}, \ddot{e}) \), the non-potential force \( Q_s = Q_s(t, q, \dot{q}) \), the electric dissipative function \( F_e = F_e(\dot{e}) \) and the dissipation functions of the viscous frictional damping force \( F_m = F_m(q, \dot{q}) \) are, respectively, transformed into \( L^* = L(t^*, q^*, \dot{q}^*, \dot{e}^*, \ddot{e}^*), \quad Q^*_s = Q_s(t^*, q^*, \dot{q}^*), \quad F^*_e = F_e(\dot{e}^*), \quad \) and \( F^*_m = F_m(q^*, \dot{q}^*). \) Expanding \( L^*, Q^*_s, F^*_e, F^*_m, \) one obtains

\[
L^* = L(t^*, q^*, \dot{q}^*, \dot{e}^*, \ddot{e}^*)
\]

\[
= L(t, q, \dot{q}, \dot{e}, \ddot{e}) + \varepsilon \left[ \frac{\partial L}{\partial t} \xi_0 + \frac{\partial L}{\partial q} \xi_s + \frac{\partial L}{\partial \dot{q}} (\dot{\xi}_s - \dot{\dot{q}}) + \frac{\partial L}{\partial \dot{e}} \eta_k + \frac{\partial L}{\partial \ddot{e}} (\ddot{\eta}_k - \ddot{\dot{e}}) \right] + O(\varepsilon^2), \quad (53)
\]

\[
Q^*_s = Q_s(t^*, q^*, \dot{q}^*) = Q_s(t, q, \dot{q}) + \varepsilon \left[ \frac{\partial Q_s}{\partial t} \xi_0 + \frac{\partial Q_s}{\partial q} \xi_s + \frac{\partial Q_s}{\partial \dot{q}} (\dot{\xi}_s - \dot{\dot{q}}) \right] + O(\varepsilon^2).
\]

(54)
\begin{equation}
F^*_e = F_e(\dot{e}^*) = F_e(\dot{e}) + \varepsilon \frac{\partial F_e}{\partial \dot{e}}(\dot{\xi}_k - \dot{\xi}_0) + O(\varepsilon^2),
\end{equation}
\begin{equation}
F^*_m = F_m(q^*, \dot{q}^*) = F_m(q, \dot{q}) + \left[ \frac{\partial F_m}{\partial q_s} \dot{\xi}_s + \frac{\partial F_m}{\partial \dot{q}_s}(\dot{\xi}_k - \dot{\xi}_0) \right] + O(\varepsilon^2).
\end{equation}

**Definition 4.** For a Lagrange mechanico-electrical systems, if physical quantities \(L(t, q, \dot{q}, e, \dot{e}), Q_s(t, q, \dot{q})\), \(F_e(\dot{e}), F_m(q, \dot{q})\) under the infinitesimal transformations (52) satisfy
\begin{equation}
\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_s} \right] - \frac{\partial L}{\partial q_s} = 0, \quad \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_k} \right] - \frac{\partial L}{\partial q_k} = \frac{\partial F^*_m}{\partial \dot{q}_s} = 0 \quad (k = 1, \ldots, m; \ s = 1, \ldots, n),
\end{equation}
the symmetry is defined as the form invariance of the systems under the infinitesimal transformation (52).

**Criterion 7.** If there exists a constant \(k\) and a gauge function \(G = G(t, q, e, \dot{e})\), and the infinitesimal generators \(\xi_0, \eta_k\) satisfy
\begin{equation}
\frac{\partial L}{\partial \dot{q}_0} + \frac{\partial L}{\partial q} \dot{\xi}_0 + \frac{\partial L}{\partial \dot{q}_k} (\dot{\xi}_k - \dot{\xi}_0) + \frac{\partial L}{\partial q} \eta_k + \frac{\partial L}{\partial \dot{q}_k} (\dot{\eta}_k - \dot{\eta}_0) = kL - \dot{G},
\end{equation}
and
\begin{equation}
\frac{d}{dt} \left[ \frac{\partial \dot{G}}{\partial \dot{q}_s} \right] - \frac{\partial \dot{G}}{\partial q_s} = 0,
\end{equation}
the symmetry is the form invariance of the Lagrange mechanico-electrical systems under the infinitesimal transformation (52).

**Proof.** Define the Euler operators of the Lagrange mechanico-electrical system as
\begin{equation}
E_s = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_s}, \quad E_k = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_k}.
\end{equation}
Using Eqs. (53) and (58), one obtains
\begin{equation}
E_s(L^*) + E_k(L^*) = E_s(L) + E_k(L) + \varepsilon (kE_s(L) + kE_k(L) - E_s(\dot{G}) - E_k(\dot{G})).
\end{equation}
Substituting Eqs. (44) and (60) into Eq. (61), one knows that the right-hand side of the Eq. (61) equals naught. Therefore
\begin{equation}
E_s(L^*) + E_k(L^*) = 0.
\end{equation}

**Definition 5.** For the Lagrange–Maxwell mechanico-electrical systems, if physical quantities \(L(t, q, \dot{q}, e, \dot{e}), Q_s(t, q, \dot{q})\), \(F_e(\dot{e}), F_m(q, \dot{q})\) under the infinitesimal transformations (52) satisfy
\begin{equation}
\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_s} \right] - \frac{\partial L}{\partial q_s} = 0, \quad \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_k} \right] - \frac{\partial L}{\partial q_k} = \frac{\partial F^*_m}{\partial \dot{q}_s} = 0 \quad (k = 1, \ldots, m; \ s = 1, \ldots, n),
\end{equation}
then the symmetry is defined as the form invariance of the systems under the infinitesimal transformations (52).

**Criterion 8.** If there exists a constant \(k\) and a gauge function \(G = G(t, q, e) = G_1(t, q, e) + G_2(t, q, e) + G_3(t, q, e) + G_4(t, q, e)\), and the infinitesimal generators \(\xi_0, \xi_s, \eta_k\) satisfy the following conditions
\begin{equation}
\frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial \dot{q}_s} (\dot{\xi}_s - \dot{\xi}_0) + \frac{\partial L}{\partial q_k} \eta_k + \frac{\partial L}{\partial \dot{q}_k} (\dot{\eta}_k - \dot{\eta}_0) = kL - \dot{G}_1,
\end{equation}
Infinite transformation

5. The connections between the Noether symmetry and the form invariance of mechanico-electrical systems

Proposition 1. For a Lagrange–Maxwell mechanico-electrical system (3), if there is a form invariance under the infinitesimal transformation (52) and there exists a gauge function \( G_N = G_N(t, q, e) \) satisfying Eq. (34), the form invariance will lead to a generalized Noether symmetry, and the system possesses a conserved quantity (49).

As a special case, there is the connections between the form invariance and the Noether symmetry of Lagrangian mechanico-electrical systems.

Proposition 2. For a Lagrange mechanico-electrical system (44), if there is the form invariance under the infinitesimal transformation (52), and there exists a gauge function \( G_N = G_N(t, q, e) \) satisfying Eq. (28), then the form invariance will lead to a Noether symmetry, and the system possesses a conserved quantity (48).
6. An example

An electrocircuit of capacitor is presented in Fig. 1. The electrode plane (mass m = 1) of capacitor is connected to a spring with elastic coefficient k. Free vibration along vertical direction is considered between electrode planks, under action of gravity, elastic and electric field force. C denotes capacitance, A denotes a constant, S denotes distance, L1 (L1 = 1) denotes unit inductance, and R denotes resistance. Power of the system is supplied by electromotive force \( E = E_0 \sin wt \). Study the Noether symmetry of the system.

Choose the equilibrium position \( (O \text{ point}) \) of active plank of capacitor as the coordinate origin, and take coordinates \( q \) and electric quantity \( e \) as generalized coordinates. The kinetic energy of the system is \( T = \frac{1}{2} \dot{q}^2 \), the spring potential energy is \( V = \frac{1}{2} kq^2 \), the magnetic energy is \( W_m = \frac{1}{2} e^2 \), and the electric energy of capacitor is \( W_e = \frac{1}{2} C \dot{e}^2 \). Then Lagrangian \( L \) of the system is

\[
L = \frac{1}{2} \dot{q}^2 + \frac{1}{2} e^2 - \frac{1}{2} kq^2 - \frac{1}{2} \frac{s - q}{A} e^2, \tag{67}
\]
dissipative function as

\[
F = F_e = \frac{1}{2} R \dot{e}^2, \tag{68}
\]
and non-conservative generalized force is

\[
u_r = E_0 \sin wt. \tag{69}
\]

Substituting Lagrange function \( L \) into generalized Killing equations (41), (42) and (43), one obtains

\[
\left( -k \dot{q} + \frac{1}{2} \dot{e}^2 \right) \frac{s - q}{A} e \eta + \left( -\frac{1}{2} \dot{q}^2 - \frac{1}{2} \dot{e}^2 - \frac{1}{2} kq^2 - \frac{1}{2} \frac{s - q}{A} e^2 \right) \left( \frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial \dot{q}} \dot{q} + \frac{\partial \xi_0}{\partial \dot{e}} \dot{e} \right) \tag{70}
\]

\[
+ \dot{q} \left( \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial \dot{q}} \dot{q} + \frac{\partial \xi}{\partial \dot{e}} \dot{e} \right) + \dot{e} \left( \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial \dot{q}} \dot{q} + \frac{\partial \eta}{\partial \dot{e}} \dot{e} \right) + (E_0 \sin wt - R \dot{e})(\eta - \dot{\xi}_0)
\]

\[
\frac{\partial G}{\partial \dot{q}} - \frac{\partial G}{\partial \dot{e}}, \tag{71}
\]

\[
\left( -\frac{1}{2} \dot{q}^2 - \frac{1}{2} \dot{e}^2 - \frac{1}{2} kq^2 - \frac{1}{2} \frac{s - q}{A} e^2 \right) \frac{\partial \xi_0}{\partial \dot{q}} + \frac{\partial \xi}{\partial \dot{q}} \dot{q} + \frac{\partial \eta}{\partial \dot{e}} \dot{e} = - \frac{\partial G}{\partial \dot{q}} \tag{72}
\]

\[
\left( -\frac{1}{2} \dot{q}^2 - \frac{1}{2} \dot{e}^2 - \frac{1}{2} kq^2 - \frac{1}{2} \frac{s - q}{A} e^2 \right) \frac{\partial \xi_0}{\partial \dot{e}} + \frac{\partial \xi}{\partial \dot{e}} \dot{q} + \frac{\partial \eta}{\partial \dot{q}} \dot{q} = - \frac{\partial G}{\partial \dot{e}}.
\]

Suppose the form of infinitesimal generators as follow

\[
\begin{align*}
\xi_0 &= C_0, \\
\dot{\xi} &= f_1(t, g, e) \dot{q} + f_2(g, e) \dot{e}, \\
\eta &= f_2(t, g, e) \dot{q} + f_3(g, e) \dot{e}. 
\end{align*} \tag{73}
\]

![Fig. 1.](image-url)
Substitution of Eq. (73) into Eq. (71) and Eq. (72) leads to
\[ \dot{q} f_1 + \dot{e} f_3 = -\frac{\partial G}{\partial q}, \quad \dot{q} f_2 + \dot{e} f_4 = -\frac{\partial G}{\partial e}. \] (74)

From Eq. (74), one gets
\[ G = \frac{1}{2} q^2 f_1 - \frac{1}{2} e^2 f_4 - \dot{q} \dot{e} f_2 + f_5(q, e), \quad f_2 = f_3. \] (75)

Substitution of Eqs. (73) and (75) into Eq. (70) yields
\begin{align*}
-kq(g_1 + f_1 \dot{q} + f_2 \dot{e}) + \frac{1}{2} e^2(g_1 + f_1 \dot{q} + f_2 \dot{e}) - \frac{s-q}{A} e(g_2 + f_2 \dot{q} + f_4 \dot{e}) \\
+ q \left( \frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial q} \dot{q} + \frac{\partial f_1}{\partial q} \dot{q}^2 + \frac{\partial f_2}{\partial q} \dot{q} \dot{e} + \frac{\partial f_1}{\partial e} \dot{e} + \frac{\partial f_2}{\partial e} \dot{e}^2 \right) \\
+ \dot{e} \left( \frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial q} \dot{q} + \frac{\partial f_2}{\partial q} \dot{q}^2 + \frac{\partial f_2}{\partial e} \dot{q} \dot{e} + \frac{\partial f_2}{\partial e} \dot{e}^2 \right) \\
+ E_0 \sin \omega t (g_2 + f_4 \dot{e} - R \dot{e}(g_2 + f_2 \dot{q} + f_4 \dot{e}) - E_0 C_0 \sin \omega t e + R C_0 \dot{e} \\
= -\frac{\partial f_5}{\partial t} + \frac{1}{2} \dot{q}^2 \frac{\partial f_1}{\partial q} + \frac{1}{2} \dot{e}^2 \frac{\partial f_4}{\partial q} + \dot{q} \dot{e} \frac{\partial f_2}{\partial q} - \frac{\partial f_5}{\partial e} + \frac{1}{2} \dot{q}^2 \frac{\partial f_1}{\partial e} + \frac{1}{2} \dot{e}^2 \frac{\partial f_4}{\partial e} + \dot{q} \dot{e} \frac{\partial f_2}{\partial e} - \frac{\partial f_5}{\partial e}. \]
\end{align*} (76)

Choosing those terms containing \( \dot{q}^3, \dot{q}^2 \dot{e}, \dot{q} \dot{e}^2, \dot{e}^3, \dot{q}^3, \dot{q}^2, \dot{q}, \dot{e}, \) one obtains the following equations
\begin{align*}
\frac{\partial f_1}{\partial q} &= \frac{1}{2} \frac{\partial f_1}{\partial q}, \\
\frac{\partial f_2}{\partial q} + \frac{1}{2} \frac{\partial f_1}{\partial q} &= \frac{\partial f_2}{\partial e} + \frac{\partial f_1}{\partial e}, \\
\frac{\partial f_2}{\partial e} + \frac{\partial f_4}{\partial q} &= \frac{1}{2} \frac{\partial f_2}{\partial q} + \frac{\partial f_4}{\partial e}, \\
\frac{\partial f_4}{\partial e} &= \frac{1}{2} \frac{\partial f_4}{\partial q}, \\
\frac{\partial g_1}{\partial q} &= 0, \\
\frac{\partial g_2}{\partial e} - R f_4 &= 0, \\
\frac{\partial g_1}{\partial e} + \frac{\partial g_2}{\partial q} - R f_2 &= 0, \\
-kq f_1 + \frac{1}{2} A e^2 f_1 - \frac{s-q}{A} e f_2 + E_0 \sin \omega t f_2 + \frac{\partial g_1}{\partial t} &= -\frac{\partial f_5}{\partial q}, \\
-kq f_2 + \frac{1}{2} A e^2 f_2 - \frac{s-q}{A} e f_4 + E_0 \sin \omega t f_4 - R g_2 - E_0 C_0 \sin \omega t + \frac{\partial g_2}{\partial t} + R C_0 &= -\frac{\partial f_5}{\partial e}, \\
-kq g_1 + \frac{1}{2} A e^2 g_1 - \frac{s-q}{A} e g_2 + E_0 \sin \omega t g_2 &= 0. \end{align*} (77)–(80)

From Eqs. (77)–(80), one has
\begin{align*}
f_1 &= -c_1 e^2 - 2 c_2 e + c_4, \\
f_2 &= c_1 q e + c_2 q + c_3 e + c_5, \\
f_4 &= -c_1 q^2 - 2 c_2 q + c_6. \end{align*} (87)
Solving Eqs. (81), (82), (83) and (73), one gets
\[ g_1 = 0, \quad g_2 = 0, \quad c_1 = 0, \quad c_2 = c_3 = c_5 = 0, \] (88)
and
\[ \xi_0 = c_0, \quad \xi = c_4 \dot{q}, \quad \eta = c_6 \dot{e}. \] (89)
Substituting Eqs. (87) and (88) into Eqs. (84) and (85), and integrating the resulting equation, one has
\[ f_s = -\frac{1}{2} k c q^2 + \frac{1}{2A} e^2 - \frac{s - q}{2A} c e^2 + E_0 c e \sin wt - E_0 c_6 e \sin wt + R c_0 e. \] (90)
Based on Eq. (75), the gauge function can be obtained
\[ G = \frac{1}{2} c q^2 - \frac{1}{2} c_6 e^2 - \frac{1}{2} k c q^2 + \frac{1}{2A} c e^2 - \frac{s - q}{2A} c e^2 + E_0 c e \sin wt - E_0 c_6 e \sin wt + R c_0 e. \] (91)
When choosing \( c_0, c_4, c_6 \) as some special value, one can obtain the following symmetries
\[ \begin{align*}
(1) & \quad c_0 = 1, \quad c_4 = c_6 = 0, \quad \xi_0 = 1, \quad \xi = \eta = 0, \quad G = -E_0 e \sin wt + Re; \\
(2) & \quad c_0 = 1, \quad c_4 = 1, \quad c_6 = 0, \quad \xi_0 = 1, \quad \xi = \dot{q}, \quad \eta = 0, \quad G = -\frac{1}{2} q^2 - \frac{1}{2} q k^2 + \frac{1}{2A} e^2 - E_0 e \sin wt + Re; \\
(3) & \quad c_0 = 1, \quad c_4 = 0, \quad c_6 = 1, \quad \xi_0 = 1, \quad \xi = 0, \quad \eta = \dot{e}, \quad G = -\frac{1}{2} \dot{e}^2 - \frac{s \dot{q}}{2A} e^2 + Re; \\
(4) & \quad c_0 = 1, \quad c_4 = 1, \quad c_6 = 1, \quad \xi_0 = 1, \quad \xi = \dot{q}, \quad \eta = \dot{e}, \quad G = -\frac{1}{2} q^2 - \frac{1}{2} q k^2 - \frac{1}{2} q k^2 + \frac{1}{2A} e^2 - \frac{s \dot{q}}{2A} e^2 + Re; \\
(5) & \quad c_0 = 0, \quad c_4 = 1, \quad c_6 = 1, \quad \xi_0 = 0, \quad \xi = \dot{q}, \quad \eta = \dot{e}, \quad G = -\frac{1}{2} q^2 - \frac{1}{2} q k^2 - \frac{1}{2} q k^2 + \frac{1}{2A} e^2 - \frac{s \dot{q}}{2A} e^2 + E_0 e \sin wt; \\
(6) & \quad c_0 = 0, \quad c_4 = 1, \quad c_6 = 0, \quad \xi_0 = 0, \quad \xi = \dot{q}, \quad \eta = 0, \quad G = -\frac{1}{2} q^2 - \frac{1}{2} k q^2 + \frac{1}{2A} e^2; \\
(7) & \quad c_0 = 0, \quad c_4 = 0, \quad c_6 = 1, \quad \xi_0 = 0, \quad \xi = 0, \quad \eta = \dot{e}, \quad G = -\frac{1}{2} \dot{e}^2 - \frac{s \dot{q}}{2A} e^2 + E_0 e \sin wt.
\end{align*} \]
Substituting the generalized quasi-symmetries (92)–(98) into Eq. (49), one obtains, respectively, the following conserved quantities
\[ \begin{align*}
I_1 & = -\frac{1}{2} \dot{q}^2 - \frac{1}{2} \dot{e}^2 - \frac{1}{2} k \dot{q}^2 - \frac{1}{2A} e^2 - E_0 e \sin wt + Re = \text{const}, \\
I_2 & = -\frac{1}{2} \dot{q}^2 - \frac{s - q}{2A} e^2 + \frac{1}{2A} e^2 - E_0 e \sin wt + Re = \text{const}, \\
I_3 & = -\frac{1}{2} \dot{q}^2 - \frac{1}{2} k \dot{q}^2 - \frac{s - q}{2A} e^2 + Re = \text{const}, \\
I_4 & = \frac{1}{2A} e^2 - \frac{s - q}{2A} e^2 + Re = \text{const}, \\
I_5 & = \frac{1}{2} \dot{q}^2 + \frac{1}{2} \dot{e}^2 - \frac{1}{2} k \dot{q}^2 + \frac{1}{2A} e^2 - \frac{s - q}{2A} e^2 + E_0 e \sin wt = \text{const}, \\
I_6 & = \frac{1}{2} \dot{q}^2 - \frac{1}{2} k \dot{q}^2 + \frac{1}{2A} e^2 = \text{const}, \\
I_7 & = \frac{1}{2} \dot{e}^2 - \frac{s - q}{2A} e^2 + E_0 \sin wt = \text{const}.
\end{align*} \]
7. Conclusion

The Noether symmetries and the form invariance are extended to mechanico-electrical systems in this Letter. The results here present significant approaches to seek conserved quantities in mechanico-electrical systems.

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References