Euler–Lagrange equation from nonlocal-in-time kinetic energy of nonconservative system

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\textbf{A B S T R A C T}

This Letter focuses on studying generalized Euler–Lagrange equation and Hamiltonian framework from nonlocal-in-time kinetic energy of nonconservative system. According to Suykens' approach, we extend his results and formulate some work related to the nonconservative system. With the Lagrangian and nonconservative force in nonlocal-in-time form, we obtain the higher order generalized Euler–Lagrange equation which leads to an extension of Newton's second law of motion. The Hamiltonian is studied in relation to the Lagrangian in the canonical phase space. Finally, the particle with nonconservative force case is studied and compared with quantum mechanical results. The extended equation gives a possible approach for understanding the connection between classical and quantum mechanics.

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1. Introduction

Nonlocal-in-time is the space–time noncommutative theory. Space–time noncommutative field theories have peculiar properties. Gomis and his partners did a lot of related work in Refs. [1–3], and they got some good results such as the Hamiltonian formalism for space–time noncommutative theories and physical degrees of freedom of nonlocal theories. The nonlocality of finite extent was got by Woodard in Ref. [4]. Llosa [5] gave the Hamiltonian formalism for nonlocal theories. Apart from these, Feynman [6] noted that the kinetic energy functional can be written as \( \frac{1}{2} \left( \sum_{k} \left( x_{k+1} - x_{k} \right) \right)^{2} \) with the position measurements of coordinates \( x \) at successive time \( t_{i+1} = t_{i} + \epsilon \). From Feynman’s conclusion, Suykens [7] started from a classical Newtonian mechanics and plugged in a nonlocal-in-time kinetic energy instead of the standard kinetic energy, and he got modification to Newtonian mechanics that could explain the quantum phenomena for the free particle case and harmonic oscillator case, but not in general. In this Letter, we will take Suykens’ approach to extend the result from conservative system to nonconservative system and find the difference between these two systems.

As remarked in [7], the kinetic energy and the nonconservative force are obtained in nonlocal form of the nonconservative system. Based on these results, we study the higher order generalized Euler–Lagrange equation which are shown to an extension of Newton’s second law of motion. The Lagrangian that we gain is singular. Following Suykens [7], in order to connect to the Ostrogradski Hamiltonian [8] of the nonconservative system, we explain the \((1 + 1)\)-dimensional formalism of nonlocal theories [1–3], which has three time coordinates with one local and one nonlocal. It can be considered as a generalization to the Ostrogradski formalism for the case of infinite derivative theories. Compared with quantum mechanics, the particle case with nonconservative force is given to gain further insight into the role of the nonlocality time extent.

This Letter is organized as follows. In Section 2 we study the idea of nonlocal-in-time kinetic energy and nonconservative force. In Section 3 we get the higher order Euler–Lagrange equation for the finite number of derivatives of nonconservative system. In Section 4, the regularization to the singular Lagrangian is made. In Section 5, the singular Lagrangian is connected to the Ostrogradski Hamiltonian. In Section 6 we discuss the Hamiltonian and nonconservative force in the \((1 + 1)\)-dimensional field theory of nonlocal
theory. In Section 7, the Hamiltonian in nonsingular higher order derivatives form is studied. Finally a simple example will be given.

2. The kinetic energy and nonconservative force in the nonlocal-in-time form

We study the system which subjects to the nonconservative force \( N \). For a nonconservative force, the work done in going from A to B depends on the path taken, such as friction, fluid resistance and air drag. We use a special nonconservative force \( N = \dot{q}^2 \), which has the same functional form with the kinetic energy. The Hamiltonian action function is defined by \( S = \int L \, dt \) where \( L = T - V \) denotes the Lagrangian containing the kinetic energy term \( T = \frac{1}{2} m \dot{q}^2 \) and the potential energy term \( V = V(q) \).

Instead of considering the standard form of kinetic energy, we use the generalized coordinates and treat the kinetic energy in nonlocal-in-time form [7] as:

\[
T_{E,n} = \frac{1}{2} m \dot{q}^2(t) \left[ \dot{q}(t + \varepsilon) + \dot{q}(t - \varepsilon) \right].
\] (1)

We take the Taylor approximations

\[
q(t + \varepsilon) \approx q(t) + \varepsilon \dot{q}(t) + \frac{\varepsilon^2}{2!} \ddot{q}(t) + \cdots + \frac{\varepsilon^n}{n!} q^{(n)}(t),
\]

\[
q(t - \varepsilon) \approx q(t) - \varepsilon \dot{q}(t) + \frac{\varepsilon^2}{2!} \ddot{q}(t) + \cdots + (-1)^n \frac{\varepsilon^n}{n!} q^{(n)}(t),
\] (2)

where \( q^{(n)}(t) \) denote the nth order time-derivatives and \( \varepsilon \) is a small positive constant and \( \varepsilon \ll t \). The interpretation of \( \varepsilon \) is here only considered at the mathematical level, and Suykens [7] gave the interpretation of the \( \varepsilon \) at the physical level.

According to [7], one gets the nonlocal-in-time kinetic energy based on the nth Taylor approximations:

\[
T_{E,n} = \frac{1}{2} m \dot{q}^2 \left[ q + \sum_{k=1}^{n} \frac{\varepsilon^k}{k!} q^{(k)}(t) \right].
\] (3)

In this way, the kinetic energy becomes

\[
T_{E,n} = \frac{1}{2} m \dot{q}^2 + \frac{1}{4} m \left[ \sum_{k=1}^{n} \frac{\varepsilon^k}{k!} q^{(k)}(t) \right],
\] (4)

and we define \( a_k \) as \( a_k = 1 + (-1)^k \), then

\[
T_{E,n} = T + \frac{1}{4} m \dot{q}^2 \left[ \sum_{k=1}^{n} a_k \frac{\varepsilon^k}{k!} q^{(k)}(t) \right].
\] (5)

With the special case \( n = 1 \), \( T_{E,1} = T \), and \( n = 2 \), \( T_{E,2} = \frac{1}{2} m \dot{q}^2 + \frac{1}{4} m \dot{q}^2 \), we denote \( L_{E,n} = T_{E,n} - V \).

Using the same methods, we get the nonconservative force \( N = \dot{q}^2 \) in the nonlocal-in-time form:

\[
N_{E,n} = \frac{1}{2} \dot{q}^2 + \frac{1}{2} \sum_{k=1}^{n} a_k \frac{\varepsilon^k}{k!} q^{(k)}(t),
\] (6)

and \( \dot{q} \) is the generalized velocity. So we get the nonconservative force based on nth Taylor approximations:

\[
N_{E,n} = \dot{q}^2 + \frac{1}{2} \sum_{k=1}^{n} a_k \frac{\varepsilon^k}{k!} q^{(k)}(t),
\] (7)

where \( a_k \) were defined before. The nonconservative force is got with the special case \( n = 1 \), \( N_{E,1} = N \) and \( n = 2 \), \( N_{E,2} = N + \frac{1}{2} \varepsilon^2 q \).

3. Higher order generalized Euler–Lagrange equation of nonconservative system

The Lagrangian \( L_{E,n} = T_{E,n} - V \) contains the higher order derivatives with \( L_{E,n}(q, \dot{q}, \ddot{q}, \ldots, q^{(m)}) \), and the nonconservative force \( N_{E,n} \) also contains the higher order derivatives with \( N_{E,n}(t, q, \dot{q}, \ddot{q}, \ldots, q^{(m)}) \), where \( Y = n + 1 \) denotes the order of the Lagrangian. Note that in relation to a Hamiltonian framework, one can consider the generalized coordinates \( q_m(t) \) that \( q_m = q_{m-1} \) and \( m = 1, 2, \ldots, Y - 1 \).

According to [9], the higher order generalized Euler–Lagrange equation of nonconservative system is:

\[
\sum_{j=0}^{Y} (-1)^j \frac{d^j}{d\varepsilon^j} \frac{\partial L_{E,n}}{\partial q^{(j)}} + N_{E,n} = 0.
\] (8)

One has

\[
\frac{\partial L_{E,n}}{\partial \dot{q}} - \frac{\partial V}{\partial q} = F,
\] (9)

where \( F \) is the conservative force. From Eq. (5), we can get

\[
\frac{\partial L_{E,n}}{\partial \dot{q}} = m \dot{q}^2 + m \sum_{k=1}^{n} a_k \frac{\varepsilon^k}{k!} q^{(k+1)},
\] (10)

which gives

\[
\frac{d}{dt} \frac{\partial L_{E,n}}{\partial \dot{q}} = m \ddot{q}^2 + m \sum_{k=1}^{n} a_k \frac{\varepsilon^k}{k!} q^{(k+2)}.
\] (11)

For \( j \geq 2 \), we can get

\[
\sum_{j=2}^{Y} (-1)^j \frac{d^j}{d\varepsilon^j} \frac{\partial L_{E,n}}{\partial q^{(j)}} = \sum_{j=2}^{Y} (-1)^{j+1} \frac{d^{j+1}}{d\varepsilon^{j+1}} \frac{\partial L_{E,n}}{\partial q^{(j+1)}} \frac{\partial q^{(j+1)}}{\partial q^{(j+1)}} \frac{\partial q^{(j+1)}}{\partial q^{(j+1)}}
\]

\[
= \sum_{j=2}^{Y} (-1)^{j+1} \frac{d^{j+1}}{d\varepsilon^{j+1}} \frac{\partial L_{E,n}}{\partial q^{(j+1)}} \frac{\partial q^{(j+1)}}{\partial q^{(j+1)}} \frac{\partial q^{(j+1)}}{\partial q^{(j+1)}}
\]

Together with Eq. (8), we get the following result:

\[
N_{E,n} + F = m \ddot{q}^2 + \frac{1}{4} m \sum_{k=1}^{n} a_k \frac{\varepsilon^k}{k!} q^{(k+2)}
\]

\[
+ \frac{1}{4} m \sum_{k=1}^{n} (-1)^k a_k \frac{\varepsilon^k}{k!} q^{(k+2)} = 0.
\] (13)

From Eq. (7), we can get Eq. (13) as follows:

\[
F - m \ddot{q} - m \sum_{k=1}^{n} \frac{\varepsilon^k}{(2k)!} q^{(2k+2)} + \ddot{q}^2 + \frac{1}{2} \dot{q}^2 \sum_{k=1}^{n} \frac{\varepsilon^k}{(2k)!} q^{(2k+1)} = 0.
\] (14)

where \( b_k = \frac{1}{2} (-1)^{k+1} a_k \), and \( b_k = 1, a_k = 2 \) when \( k \) is even or \( b_k = 0, a_k = 0 \) when \( k \) is odd.

So for \( n \) the generalized Euler–Lagrange equation of nonconservative system is

\[
F - m \ddot{q} - \frac{n}{2} \sum_{k=1}^{n/2} \frac{\varepsilon^{2k}}{(2k)!} q^{(2k+2)} + \ddot{q}^2 + \ddot{q}^2 \sum_{k=1}^{n/2} \frac{\varepsilon^{2k}}{(2k)!} q^{(2k+1)} = 0.
\] (15)

where only the even derivatives terms remain. For \( \varepsilon = 0 \), we obtain the equation of motion:

\[
F - m \ddot{q} + q^2 = 0.
\] (16)
and this result is the classical Newton’s law of motion of the non-conservative system. If \( n = 2, \ k = 1 \), we can also get the special result:

\[
F - m\ddot{q} - m\frac{e^2}{2!}q^{(4)} + \dot{q}^2 + \ddot{q}^2 = 0. \tag{17}
\]

Making use of the Taylor approximations (2) and (6), in the limit case \( n \to \infty \), we get the extended equation

\[
F - m\frac{1}{2}\left[\dot{q}(t + \varepsilon) + \dot{q}(t - \varepsilon)\right] + \frac{1}{2}\left[\dot{q}(t + \varepsilon) + \dot{q}(t - \varepsilon)\right] = 0. \tag{18}
\]

4. A regularized Lagrangian of the nonconservative system

For a Lagrangian of order \( Y \) to be regular, the equation of motion contains \( q^{(2Y)} \) and determines \( q^{(2Y)} \) as a function of \( q \) and its \( 2Y - 1 \) derivatives [4]. Hence the Lagrangian \( L_{e,n} \) is singular when the highest term is \( n = Y - 1 \), and a regular Lagrangian can be considered that keeps the same form of equations, completed with additional terms. This is done by taking the following regularization when \( n \) even [7]:

\[
L_{e,n}^{\text{reg}} = L_{e,n} + R_{e,n} \tag{19}
\]

with the regularization part

\[
R_{e,n} = \frac{1}{2} \sum_{j=Y-k_Y}^{Y} (-1)^{j+1} \tau_j (q^{(j)})^2, \tag{20}
\]

where \( \tau_j = m\frac{e^{j-2}}{(2j)!} \) and \( k_Y = \frac{Y-2}{2} \) (when \( k_Y \) is negative, it means there is no regularization).

Then the higher order generalized Euler–Lagrange equation of nonconservative system with the additional terms is:

\[
\sum_{j=Y-k_Y}^{Y} (-1)^{j+1} \frac{d}{dt} \frac{\partial R_{e,n}}{\partial q^{(j)}} = \sum_{j=Y-k_Y}^{Y} (-1)^{j+1} \tau_j q^{(j)}.
\]

Then the higher order generalized Euler–Lagrange equation of nonconservative system turns into:

\[
F - m\ddot{q} - m\sum_{k=1}^{n/2} \frac{e^{2k}}{(2k)!} q^{(2k+2)} - \sum_{j=Y-k_Y}^{Y} \tau_j q^{(j)} = 0.
\]

According to [7], we obtain the higher order generalized Euler–Lagrange equation with regularized Lagrangian of the nonconservative system:

\[
F - m\ddot{q} - m\sum_{k=2}^{n/2} \frac{e^{2k-2}}{(2k-2)!} q^{(2k)} + \dot{q}^2 + \ddot{q}^2 = 0. \tag{22}
\]

which leads to the same limits (18) for \( n \to \infty \).

For the case \( n = 2, Y = 3 \), we get as special case:

\[
F - m\ddot{q} - m\frac{e^2}{2!}q^{(4)} - m\frac{e^4}{2!}q^{(6)} + \dot{q}^2 + \ddot{q}^2 = 0. \tag{24}
\]

5. Ostrogradski Hamiltonian of the nonconservative system

The regularized Lagrangian equation (19) is nondegenerate. And the equation of motion contains \( q^{(2Y)} \) as a function of \( q \) and its \( 2Y - 1 \) derivatives, we can denote \( Y \) coordinates and \( Y \) conjugate momenta in the canonical phase space [4].

In Ostrogradski framework the \( M \)th coordinate is just the \((M-1)\)th derivatives of \( q \)

\[
Q_{M} = q^{(M-1)}. \tag{25}
\]

The momentum canonically conjugate to \( Q_{M} \) is

\[
P_{M} = \sum_{j=M}^{Y} \left( -\frac{d}{dt} \frac{\partial L_{e,n}^{\text{reg}}}{\partial q^{(j)}} \right).
\]

A consequence of nondegeneracy is that the derivatives \( q^{(Y+M)} \) can be determined from \( P_{Y-M}, P_{Y-M+1}, \ldots, P_{Y} \) and \( Q_{M} \). From these new coordinates and momenta, we can get nonconservative force \( N_{e,n} = N_{e,n}(t, Q_1, Q_2, \ldots, Q_Y, P_1, P_2, \ldots, P_{Y}) \) and the Lagrangian \( L_{e,n}^{\text{reg}} = L_{e,n}(t, Q_1, Q_2, \ldots, Q_Y, P_1, P_2, \ldots, P_{Y}) \) in canonical form.

The Ostrogradski Hamiltonian is:

\[
H = \sum_{M=1}^{Y} P_{M} \dot{Q}_{M} - L_{e,n}^{\text{reg}}. \tag{27}
\]

It’s straightforward to examine that the various canonical evolution equations reproduce the equation of motion and the structure of the canonical formalism: \( Q_{M} \) gives the canonical definition (25) for \( Q_{M-1} \), \( P_{M+1} \) gives the canonical definition (26) for \( P_{M} \). So there is no doubt that Ostrogradski’s Hamiltonian generates time evolution.

6. (1 + 1)-dimensional field theory of nonlocal theory

As discussed in [1], we define a time evolution \( T_{1} \) for a given initial trajectory \( q(\lambda) \), and introduce new dynamical variables \( Q(t, \lambda) \), which satisfy

\[
Q(t, \lambda) = q(\lambda + t). \tag{28}
\]

Condition (28) in differential form should be:

\[
\dot{Q}(t, \lambda) = \dot{Q}'(t, \lambda) = \frac{\partial}{\partial \lambda} Q(t, \lambda), \tag{29}
\]

where the overdot stands for \( \dot{\lambda} \) and prime stands for \( \partial_{\lambda} \).

The Hamiltonian of the nonconservative system becomes:

\[
H[t, \{Q, P\}] = \int \left[ P(t, \lambda) Q'(t, \lambda) - \dot{L}[t, \{Q\}] \right] d\lambda. \tag{30}
\]

where \( P \) is the canonical momentum. The nonconservative force becomes \( N_{e,n} = N_{e,n}(t, Q(t, \lambda), P(t, \lambda)) \). And the new canonical variables with Poisson brackets become:

\[
\delta(\lambda - \lambda') \].

In Eq. (30), \( \dot{L}(t, \{Q\}) \) can be defined as a function [1]

\[
\dot{L}(t, \{Q\}) = \int \delta(\lambda) \delta(t, \lambda) d\lambda. \tag{32}
\]

So the Hamiltonian (30) becomes

\[
H[t, \{Q, P\}] = \int \left[ P(t, \lambda) \frac{\partial}{\partial \lambda} Q(t, \lambda) - \delta(\lambda) \delta(t, \lambda) \right] d\lambda. \tag{33}
\]

\( \delta \) is the Lagrangian density which is built from the nonlocal Lagrangian \( L_{e,n}^{\text{non}} \) by replacing \( q(t) \to Q(t, \lambda), \frac{d}{dt} q(t) \to \frac{d}{dt} Q(t, \lambda) \), and \( q(t + \varepsilon) \to Q(t, \lambda + \varepsilon) \). Based on this formalism, a \((1 + 1)\) field theory contains two time coordinates \( t \) and \( \lambda \), and is local in \( t \) and nonlocal in \( \lambda \).
7. Hamiltonian in nonsingular higher order derivate form

We would like to derive the Hamiltonian in nonsingular higher order derivate form from the nonlocal theories. For \( n \to \infty \) and a regular Lagrangian with the \( L^{m,n} \), the higher order theory can be embedded in the nonlocal form with the infinite-dimensional phase space with Taylor basis [1]:

\[
Q(t, \lambda) = \sum_{n=0}^{\infty} e_n(\lambda) q^n(t),
\]

\[
P(t, \lambda) = \sum_{n=0}^{\infty} e^n(\lambda) p_n(t),
\]

where \( e_n \) and \( e^n \) are orthonormal bases,

\[
e_n(\lambda) = \frac{\lambda^n}{n!}, \quad e^n(\lambda) = (-\partial_n)^n \delta(\lambda)
\]

and \( P(t, \lambda) \) is the canonical momentum of \( Q(t, \lambda) \). With these new Taylor bases, we get the nonconservative force \( N_{\lambda,n}(t, Q(t, \lambda), P(t, \lambda)) \). The coefficients \( q^n(t) \) and \( p^n(t) \) are the canonical variables with Poisson brackets

\[
\{q^m(t), p^n(t)\} = \delta^m_n.
\]

In this way, the Hamiltonian becomes

\[
H(t) = \sum_{n=0}^{\infty} p_n(t) q^{n+1}(t) - L^{m,n}(q^0, q^1, \ldots, q^n).
\]

8. Particle case with the nonconservative force

We get the Euler–Lagrange equation from nonlocal-in-time kinetic energy and related theories of the nonconservative system. Compared with the holonomic conservative system subject to the conservative force, we study the holonomic nonconservative system which not only subjects to the conservative force but also the nonconservative force. Apart from this, one can get that the Euler–Lagrange equation of the nonconservative system is an inhomogeneous equation, and the related theories changed.

Now let’s get the particle (\( F = 0 \)) for Eq. (15):

\[
\ddot{q} + \sum_{k=1}^{n/2} \frac{\varepsilon^2 k}{(2k)!} q^{(2k+1)} = -m \ddot{\eta} - m \sum_{k=1}^{n/2} \frac{\varepsilon^2 k}{(2k)!} q^{(2k+2)} = 0.
\]

We propose a solution of the form \( q(t) = e^{i\omega t} \), which gives Eq. (38):

\[
\omega^2 \left( 1 - \frac{\varepsilon^2}{2!} \omega^2 + \frac{\varepsilon^4}{4!} \omega^4 - \cdots \right) e^{i\omega t} = 0.
\]

So Eq. (39) is a finite \( n \) approximation to

\[
\omega^2 \cos(\varepsilon \omega) e^{i\omega t} - \omega^2 \cos(\varepsilon \omega) e^{2i\omega t} = 0.
\]

Eq. (40) is satisfied for \( \omega = 0 \) or \( \omega = \frac{(2l+1)\pi}{\varepsilon} \) with \( l \in \mathbb{Z} \). So the solution of the form \( q(t) = c_1 + c_2 t + \sum_{j=1}^{\infty} c_{j,1} \cos(\omega_0 t) + \sum_{j=1}^{\infty} c_{j,2} \sin(\omega_0 t) \) with unknown coefficients \( c_1, c_2, c_{j,1}, c_{j,2} \). Here the oscillatory modes are superimposed on the straight line motion. Starting from (18)

\[
-m \frac{1}{2} \left[ \ddot{q}(t + \varepsilon) + \ddot{q}(t - \varepsilon) \right] + \frac{1}{2} \left[ q(t + \varepsilon) + q(t - \varepsilon) \right] = 0,
\]

leads to the same result, and we propose a solution of the form \( q(t + \varepsilon) = A(t) \varepsilon B(t) \) with \( A(t) = e^{i\omega t} \) and \( B(t) = e^{i\omega t} \), which has the special case for \( q(t) = A(t) \varepsilon B(0) \) where \( B(0) = 1 \). So we can get

\[
\omega^2 \frac{1}{2} [B(t + \varepsilon) - B(t)] A(t) - \omega^2 \frac{1}{2} [B(t + \varepsilon) + B(t)] A^2(t) = 0.
\]

with \( \tilde{A}(t) = -\omega^2 A(t) \), \( \tilde{A}^2(t) = -\omega^2 \tilde{A}^2(t) \) and \( \frac{1}{2} [B(t + \varepsilon) + B(t)] = \cos(\omega_0) \). For the mode \( \omega_0 = 0 \), the solution is then got by \( \tilde{A}(t) = 0 \), \( \tilde{A}^2(t) = 0 \), which gives the classical straight line motion. And the nonzero \( \omega_0 \) leads to classical harmonic oscillator solutions \( \tilde{A}(t) = -\omega^2 A(t) \).

We can also write the mode as follows:

\[
m D^2 \cos(\varepsilon D)Q(t) - D^2 \cos(\varepsilon D)Q^2(t) = 0,
\]

where \( D \) is the time derivative operator. From [10] the following product representation exists,

\[
\cos(\varepsilon D) = \prod_{i=1}^{\infty} \left( 1 + \frac{D^2}{\alpha^2} \right),
\]

in which the factors correspond to quanta. From the above discussion of the nonconservative system, the coefficients and other solutions are identical to the free particle case in Suykens [7] because of choosing the special nonconservative force which has the same functional form with the kinetic energy.

9. Conclusion

The higher order generalized Euler–Lagrange equation and Hamiltonian framework of the nonconservative system has been given with the nonlocal-in-time Lagrangian and nonconservative force. We can use the same way to extend the nonholonomic system for future studies.

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