Envelope compacton and solitary pattern solutions of a generalized nonlinear Schrödinger equation

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Abstract

In this paper, the generalized nonlinear Schrödinger equation with nonlinear dispersion \(iu_t + a(u|u|^{n-1})_{xx} + bu|u|^{m-1} = 0\) (called the GNLS\((m, n)\) equation) is investigated by using the theory of a dynamical system. As a result, we obtain some envelope compacton and solitary pattern solutions of the GNLS\((m, n)\) equation. In addition, we point out the reason for the appearance of the unsmooth solutions.

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1. Introduction

The well-known cubic nonlinear Schrödinger equation

\[iu_t + au_{xx} + bu|u|^2 = 0\]  \hspace{1cm} (1.1)

which arises from the study of nonlinear wave propagation in dispersive and inhomogeneous media is widely used in basic models of nonlinear waves in many areas of physics. It is a generic equation describing the evolution of the slowly varying amplitude of a nonlinear wave train in weakly nonlinear, strongly dispersive, and hyperbolic systems [3]. Furthermore, it is well known that the propagation of picosecond pulses in optical fibers is described by the nonlinear Schrödinger equation [4,8,9]. A considerable amount of research work has been devoted to the study of nonlinear Schrödinger equations and their generalizations [10,11,14]. Many numerical or analytical methods such as inverse scattering methods [1], Hirota bilinear forms [2], tanh methods [7] and sine–cosine methods [10,11], dynamical system methods [5,6,12,13] have been successfully applied to solve the nonlinear evolution equations.

In 1993, a new type of soliton with compact support, or strict localization of solitary waves, which was named the compacton appeared in the work of Rosenau and Hyman [9]. Recently, in an effort to understand the role of nonlinear
dispersion in complex nonlinear wave equations, Yan [11] introduced and studied a nonlinear Schrödinger equation with nonlinear dispersion, described as

\[ iu_t + (u|u|^{n-1})_{xx} + \mu u|u|^{m-1} = 0. \]  

(1.2)

In this paper, we generalize the nonlinear Schrödinger equation to a more extensive one as follows:

\[ iu_t + a(u|u|^{n-1})_{xx} + bu|u|^{m-1} = 0 \]

(1.3)

which we call the GNLS\((m, n)\) equation.

Clearly, when \(n = 1\) and \(m = 3\), (1.3) is the nonlinear Schrödinger equation (1.1) and when \(a = 1\), it is just the NLS\((m, n)\) equation of Yan [11].

Under the transformation \(u(x, t) = U(\xi) \exp(i\sigma t), \xi = kx\), where \(k, \sigma\) are real parameters and \(U(\xi)\) is a nonnegative real valued function, Eq. (1.3) is reduced to the following nonlinear ordinary differential equation:

\[ -\sigma U + ak^2 \frac{d^2(U^n)}{d\xi^2} + bu^m = 0. \]  

(1.4)

When \(m = n = 1\), (1.4) is a simple linear differential equation. We only consider the solutions of the GNLS\((m, n)\) equation when \(m = n\) or \(n = 1\) in this paper.

The rest of this paper is arranged as follows: In Section 2, we study the compacton and solitary pattern solutions of the GNLS\((m, 1)\) equation. The compacton and solitary pattern solutions of the GNLS\((n, n)\) equation are investigated in Section 3.

2. Solutions of the GNLS\((m, 1)\) equation

When \(n = 1\) and \(m \neq 1\), letting \(V = U^{m-1}\) and inserting it in Eq. (1.4), we have

\[ -\sigma + \frac{ak^2}{m-1} \left[ V^{-2} V'' + \frac{2-m}{m-1} V^{-2} V^{'2} \right] + bV = 0 \]  

(2.1)

when \(V \neq 0\).

Let \(V' = y\); then Eq. (2.1) is equivalent to the following two-dimensional system:

\[ V' = y, \quad y' = \frac{m-2}{m-1} \frac{y^2}{V} - \frac{m-1}{ak^2} V[bV - \sigma] \]

(2.2)

which has the first integral as follows:

\[ H(V, y) = \frac{1}{2} V^{\frac{2}{m-1}} \left[ y^2 + \frac{(m-1)^2}{ak^2} \left( \frac{b}{m+1} V^3 - \frac{\sigma}{2} V^2 \right) \right] = h \]

(2.3)

when \(m \neq -1\). Note that \(V = 0\) is a straight line in the \((V, y)\)-phase plane of the system (2.2).

From (2.3) we get

\[ y = \pm \frac{m-1}{k} V \sqrt{\frac{2k^2}{(m-1)^2} hV^{\frac{2}{m-1}} + \frac{\sigma}{a m+1} - \frac{2b}{a(m+1)} V}. \]  

(2.4)

Inserting Eq. (2.4) into the first equation of (2.2), it follows that

\[ \frac{dV}{V \sqrt{\frac{k^2}{(m-1)^2} hV^{\frac{-2}{m-1}} + \frac{\sigma}{a m+1} - \frac{2b}{a(m+1)} V}} = \pm \frac{m-1}{k} d\xi. \]  

(2.5)

Obviously, the solutions of the GNLS equation (1.3) can be obtained by using the integral of Eq. (2.5). Next, we will study the envelope compactons and solitary pattern solutions of the GNLS equation by integrating Eq. (2.5) and solving \(V\).
Letting $h = 0$, Eq. (2.5) is reduced to
\[
\frac{dV}{V \sqrt{\frac{a}{a} - \frac{2b}{(m+1)a}} V} = \pm \frac{m-1}{k} d\xi.
\] (2.6)

Notice that $V = 0$ is a straight line in the $(V, y)$-phase plane of the system (2.2). Suppose that $\xi_0 = kx_0$ is a constant which satisfies $V(\xi_0) = \frac{(m+1)a}{2b}$. For different values of the parameters, we obtain the solutions of Eq. (2.1) from the integral of Eq. (2.6) as follows:

1. For $a \sigma < 0$, Eq. (2.1) has an explicit solution
\[
V = \frac{(m+1)\sigma}{2b} \sec^2 \left( \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} (\xi - \xi_0) \right)
\] (2.7) with
\[
\left| \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} (\xi - \xi_0) \right| < \frac{\pi}{2}.
\]
2. For $a \sigma > 0$, Eq. (2.1) has an explicit solution
\[
V = \frac{(m+1)\sigma}{2b} \sec^2 h \left( \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} (\xi - \xi_0) \right)
\] (2.8) and
\[
V = -\frac{(m+1)\sigma}{2b} \csc^2 h \left( \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} \xi \right)
\] (2.9) with $\xi \neq 0$.

From the above analysis and the formulas (2.7)-(2.9), we obtain the envelope compacton and solitary pattern solutions of the GNLS$(m, 1)$ equation which we describe in the following theorem.

**Theorem 1.** The generalized nonlinear Schrödinger equation with nonlinear dispersion when $n = 1$, i.e. the GNLS$(m, 1)$ equation, has solutions described as follows.

1. When $a \sigma < 0$, $m < 1$ and $\frac{2b}{(m+1)a} > 0$
\[
u(x, t) = \begin{cases} \left[ \frac{2b}{(m+1)\sigma} \cos^2 \left( \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} (x - x_0) \right) \right]^{\frac{1}{m}} e^{i\sigma t} & | \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} (x - x_0) | < \frac{\pi}{2} \\ 0 \end{cases}
\] (2.10)

is an envelope compacton pattern solution of the GNLS$(m, 1)$ equation.

2. When $a \sigma > 0$, $\frac{(m+1)\sigma}{2b} > 0$
\[
u(x, t) = \left[ \frac{(m+1)\sigma}{2b} \sec^2 \left( \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} x \right) \right]^{\frac{1}{m-1}} e^{i\sigma t}
\] (2.11)

is an envelope solitary pattern solution of the GNLS$(m, 1)$ equation which is bounded when $m > 1$.

3. When $a \sigma > 0$ and $\frac{(m+1)\sigma}{2b} < 0$
\[
u(x, t) = \left[ -\frac{(m+1)\sigma}{2b} \csc^2 \left( \frac{m-1}{2k} \sqrt{\frac{\sigma}{a}} x \right) \right]^{\frac{1}{m-1}} e^{i\sigma t}
\] (2.12)

with $x \neq 0$ is an unbounded envelope solitary pattern solution of the GNLS$(m, 1)$ equation.

3. **Solutions of the GNLS$(n, n)$ equation**

When $n \neq 1$, letting $W = U^{n-1}$ and inserting it in Eq. (1.4), we easily derive the following system:
\[
W' = z, \quad z' = -\frac{1}{n-1} \frac{z^2}{W} - \frac{n-1}{ak^2n} \left[ bW^{\frac{m-1}{n}} - \sigma \right]
\] (3.1)
which has the first integral as follows:

$$H(W, y) = \frac{1}{2} W^{2-\alpha} \left[ z^2 + 2 \frac{(n-1)^2}{ak^2n} \left( \frac{b}{m+n} W^{m+n} - \frac{\sigma}{n+1} W \right) \right] = h.$$  \hspace{1cm} (3.2)

Note that $W = 0$ is a straight line in the $(W, y)$-phase plane of the system (3.1).

From (3.2) we get

$$z = \pm \frac{n-1}{k} \sqrt{-\frac{2}{na} \left( \frac{b}{m+n} W^{m+n} - \frac{\sigma}{n+1} W \right) + \frac{2k^2 h}{(n-1)^2} W^{-\frac{2}{n+1}}}. \hspace{1cm} (3.3)$$

Inserting Eq. (3.3) into the first equation of (3.1), it follows that

$$\frac{dW}{\sqrt{-\frac{2}{na} \left( \frac{b}{m+n} W^{m+n} - \frac{\sigma}{n+1} W \right) + \frac{2k^2 h}{(n-1)^2} W^{-\frac{2}{n+1}}}} = \pm \frac{n-1}{k} d\xi. \hspace{1cm} (3.4)$$

Obviously, the solutions of Eq. (3.1) can be derived by using the integral of Eq. (3.4). When $m = n$, letting $h = 0$, Eq. (3.4) is reduced to

$$\frac{dW}{\sqrt{-\frac{b}{n} W^2 + \frac{2n\sigma}{n+1} W}} = \pm \frac{n-1}{nk} d\xi. \hspace{1cm} (3.5)$$

For different values of the parameters, we obtain the solutions of system (3.1) from the integral of Eq. (3.5) as follows:

1. For $ab < 0$, (3.1) has an explicit solution

$$W(\xi) = \frac{2a\sigma}{b(n+1)} \cosh^2 \left( \frac{n-1}{2nk} \sqrt{-\frac{b}{a} (\xi - \xi_0)} \right),$$  \hspace{1cm} (3.6)

$$W(\xi) = -\frac{2a\sigma}{b(n+1)} \sinh^2 \left( \frac{n-1}{2nk} \sqrt{-\frac{b}{a} (\xi - \xi_0)} \right).$$  \hspace{1cm} (3.7)

2. For $ab > 0$, (3.1) has an explicit solution

$$W(\xi) = \frac{2a\sigma}{b(n+1)} \cos^2 \left( \frac{n-1}{2nk} \sqrt{\frac{b}{a} (\xi - \xi_0)} \right). \hspace{1cm} (3.8)$$

with $\left| \frac{n-1}{2nk} \sqrt{\frac{b}{a} (\xi - \xi_0)} \right| < \frac{\pi}{2}$.

From the above analysis and the formulas (3.6)–(3.8), we obtain the envelope compacton and solitary pattern solutions of the GNLS$(n, n)$ equation which we describe in the following theorem.

**Theorem 2.** The nonlinear Schrödinger equation with nonlinear dispersion when $m = n$, i.e. the NLS$(n, n)$ equation, has the solutions described as follows.

1. When $ab > 0$, $n > 1$ and $\sigma > 0$

$$u(x, t) = \begin{cases} 
\left[ \frac{2a\sigma}{b(n+1)} \cos^2 \left( \frac{n-1}{2n} \sqrt{\frac{b}{a} (x - x_0)} \right) \right]^\frac{1}{\sigma} e^{i\sigma t} \left| \frac{n-1}{2n} \sqrt{\frac{b}{a} (x - x_0)} \right| < \frac{\pi}{2} \\
0 \text{ otherwise}
\end{cases} \hspace{1cm} (3.9)$$

is a compacton pattern solution of the GNLS$(n, n)$ equation.

2. When $ab < 0$ and $\frac{\sigma}{(n+1)} < 0$,

$$u(x, t) = \left[ \frac{2a\sigma}{b(n+1)} \cos^2 \left( \frac{n-1}{2nk} \sqrt{-\frac{b}{a} (x - x_0)} \right) \right]^\frac{1}{\sigma} e^{i\sigma t} \hspace{1cm} (3.10)$$

$$\text{with } \left| \frac{n-1}{2nk} \sqrt{-\frac{b}{a} (x - x_0)} \right| < \frac{\pi}{2}.$$
is a solitary pattern solution of the GNLS\((n, n)\) equation. When \(n < 1\), (3.10) is a bounded solution.

3. When \(ab < 0\) and \(\frac{na}{n+1} > 0\),

\[
u(x, t) = \left[ -2an\sigma \frac{b(n + 1)}{b(n + 1)} \sinh^2 \left( \frac{n - 1}{2n} \sqrt{\frac{b}{a}} x \right) \right]^{\frac{1}{n-1}} e^{\sigma t}
\]

with \(x \neq 0\) is a solitary pattern solution of the GNLS\((n, n)\) equation.

**Remark.** We present a new method different from the sin–cos method and sinh–cosh method for seeking the envelope compacton and solitary pattern solutions of a nonlinear Schrödinger equation with nonlinear dispersion. It is easy to establish that the results obtained in this paper include not only some results of Yan [11] as special cases but also some exact envelope compacton and solitary pattern solutions of a nonlinear Schrödinger equation with nonlinear dispersion in a more generalized form.

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