DYNAMIC STABILITY OF AN AXIALLY ACCELERATING VISCOELASTIC BEAM WITH TWO FIXED SUPPORTS

X.-D. YANG

Department of Engineering Mechanics
Shenyang Institute of Aeronautical Engineering
Shenyang 110034, China

L.-Q. CHEN*

Department of Mechanics, Shanghai University
Shanghai 200072, China
Shanghai Institute of Applied Mathematics and Mechanics
Shanghai University, Shanghai 200072, China
lqchen@staff.shu.edu.cn

Received 25 March 2005
Accepted 11 November 2005

The dynamic stability of an axially accelerating viscoelastic beam with two fixed supports is investigated. The Kelvin model is used for the constitutive law of the beam. A small simple harmonic is allowed to fluctuate about the constant mean speed applied to the beam, and the governing equation is truncated using the Galerkin method based on the eigenfunctions of the stationary beam. The averaged equations are derived for the cases of subharmonic and combination resonance. Finally, numerical examples are presented to demonstrate the effects of the viscosity coefficient, the mean axial speed and the beam bending stiffness on the stability boundaries.

Keywords: Axially accelerating beam; stability; viscoelastic beam; Galerkin truncation; averaging method.

1. Introduction

Many engineering devices, such as band saws and serpentine belts, may be simulated as axially moving beams. Despite its wide applications, the axially moving beam suffers from the occurrence of large transverse vibrations due to variations in axial speed. Axial transport acceleration frequently appears in engineering systems. Pasin studied the stability of transverse vibrations of beams with periodically reciprocating motion in axial direction.1 Öz, Pakdemirli and ÖzKayası applied the method of multiple scales to study the dynamic stability of an axially accelerating beam with small bending stiffness.2 ÖzKayası and Pakdemirli employed the

*Corresponding author.
method of multiple scales and the method of matched asymptotic expansions to construct non-resonant boundary layer solutions for an axially accelerating beam with small bending stiffness. Öz and coworkers applied the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating tensioned beam under simply supported conditions and fixed conditions. Parker and Lin adopted a one-term Galerkin discretization and the perturbation method to study the dynamic stability of an axially accelerating beam subjected to a tension fluctuation. Recently, Özkan and Öz applied artificial neural network algorithm to determine stability of an axially accelerating beam.

All the above-mentioned researchers assumed the beam to be elastic with no dissipative mechanisms. However, it was noted that the effect of dissipative mechanisms is important for materials with axial vibration, which can be very well modeled as viscoelastic materials. The literature is quite limited on the subject of concern here. Based on a two-term Galerkin truncation, Chen and Yang analyzed the stability of axially accelerating beams with simply support ends and fixed ends, and studied numerically the bifurcation and chaos of an axially accelerating nonlinear beam.

In the present study, the method of averaging is applied to analyzing the dynamic stability of an axially moving viscoelastic beam with fixed ends. The governing equation is truncated using the Galerkin method. The stability conditions are derived for subharmonic and combination resonance from the averaged equation. Finally, numerical simulations are presented to demonstrate the effects of the viscosity coefficient, the mean axial speed and the beam bending stiffness.

2. Galerkin Truncation of the Governing Equation

A uniform viscoelastic beam, with linear density $\rho$, cross-sectional area $A$, moment of inertia $I$ and initial tension $P$, travels along the axial direction at the time-dependent speed $V(T)$ between two ends of distance $L$. The constitutive law of the beam is defined by the Kelvin model with the elastic constant $E$ and viscosity coefficient $\alpha$. Considering only the bending vibration, in terms of the transverse displacement $W(X, T)$, where $T$ is time, and $X$ is the axial coordinate, the governing equation is given by

$$\rho A(\ddot{W} + 2V\dot{W}' + VW' + V^2W'') - PW'' + EIW^{(4)} + \alpha IW^{(4)} = 0,$$

which can be cast into a nondimensional form as

$$\dot{w} + 2\gamma \dot{w}' + \dot{\gamma}w' + (\gamma^2 - 1)w'' + \beta^2 w^{(4)} + \kappa w^{(4)} = 0$$

with the parameters defined as

$$x = \frac{X}{L}, \quad w = \frac{W}{L}, \quad t = \frac{T}{\sqrt{P/A\rho}}, \quad \gamma = V \sqrt{\frac{Ap}{P}}, \quad \beta^2 = \frac{EI}{PL^2}, \quad \kappa = \frac{\alpha A}{\sqrt{PA\rho}}.$$  

Here, $\kappa$ is a small quantity, if the viscosity is considered rather weak.
For the stationary beam with two fixed ends, the eigenfunctions are\(^{13}\)

\[
\Phi_i(x) = \cos \psi_i x - \cosh \psi_i x - \frac{\cos \psi_i - \cosh \psi_i}{\sin \psi_i - \sinh \psi_i} (\sin \psi_i x - \sinh \psi_i x),
\]

(4)

where \(\psi_i\) is the \(i\)th root of the frequency equation

\[
\cos \psi_i \cosh \psi_i - 1 = 0.
\]

(5)

Then the Galerkin method is used to simplify Eq. (2). First of all, the solution of Eq. (2) may be given in by an expanded series as

\[
w(x, t) = \Phi^T q,
\]

(6)

where

\[
q = (q_1(t) q_2(t) \cdots q_N(t))^T
\]

\[
\Phi = (\Phi_1(x) \Phi_2(x) \cdots \Phi_N(x))^T.
\]

Substituting Eq. (6) into Eq. (2), multiplying by \(\Phi\), and integrating over \([0, 1]\) lead to the Galerkin discretization of the governing equation:

\[
\ddot{q} + 2\gamma B_1 \dot{q} + \gamma B_2 q + (\gamma^2 - 1) B_2 q + \beta^2 \Lambda q + \kappa \Lambda \dot{q} = 0,
\]

(8)

where

\[
B_1 = \int_0^1 \Phi \Phi'^T \, dx \quad B_2 = \int_0^1 \Phi \Phi''^T \, dx \quad \Lambda = \text{diag}\{\psi_1^4 \psi_2^4 \cdots \psi_N^4\}.
\]

(9)

3. Formulation for the Averaging Analysis

The transport speed is assumed as a simple harmonic variation about the constant mean speed, i.e.

\[
\gamma = \gamma_0 + \gamma_1 \sin(\omega t),
\]

(10)

where \(\gamma\) is a small quantity of the same order as \(\kappa\). Substituting Eq. (10) into Eq. (8) and neglecting higher order terms yield

\[
\dot{y} = Ay + \gamma_1 A_1 y \sin \omega t + \omega \gamma_1 A_2 y \cos \omega t + \kappa A_3 y,
\]

(11)

where

\[
y = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ - (\gamma_0^2 - 1) B_2 - \beta^2 \Lambda & -2\gamma_0 B_1 \end{pmatrix}
\]

\[
A_1 = \begin{pmatrix} 0 & 0 \\ -2\gamma_0 \Lambda & -2B_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -B_1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\Lambda \end{pmatrix}.
\]

(12)

For \(\gamma_1 = 0\) and \(\kappa = 0\), Eq. (11) defines a gyroscopic system whose eigenvalues are all imaginary due to truncation of the higher order terms,

\[
\ddot{\omega} + 2\gamma_0 \omega' + (\gamma_0^2 - 1) \omega'' + \beta^2 \omega^{(4)} = 0.
\]

(13)
Therefore, there exists a canonical transformation matrix $T$ such that

$$T^T A T = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_N \end{pmatrix},$$

(14)

where

$$\lambda_n = \begin{pmatrix} 0 & -\omega_n \\ \omega_n & 0 \end{pmatrix}.$$ 

(15)

By introducing the state variable transformation,

$$x = Ty,$$

(16)

Eq. (11) can be expressed in terms of the new state variables as

$$\dot{x} = \Omega x + \gamma_1 Cx \sin \omega t + \gamma_1 \omega Dx \cos \omega t + \kappa Ex,$$

(17)

where

$$\Omega = T^T A T \quad C = T^T A_1 T \quad D = T^T A_2 T \quad E = T^T A_3 T.$$ 

(18)

Consider the speed frequency $\omega$ fluctuating about a given value $\omega_0$ with a small variation $\Delta$, that is,

$$\omega = \omega_0 + \Delta.$$ 

(19)

A new time variable is introduced as

$$\tau = \omega t.$$ 

(20)

The substitution of Eqs. (19) and (20) into Eq. (17) and neglecting the higher order terms yields

$$\dot{x} = \frac{\Omega x}{\omega_0} - \frac{\Delta}{\omega_0^2} \Omega x + \frac{\gamma_1}{\omega_0} Cx \sin \tau + \frac{\gamma_1}{\omega_0} Dx \cos \tau + \frac{\kappa}{\omega_0} Ex.$$

(21)

Rewriting $x$ in an amplitude-phase form

$$x_{2n-1} = a_n \cos \phi_n, \quad x_{2n} = a_n \sin \phi_n \quad (n = 1, 2, \ldots, N),$$

(22)

where

$$\phi_n = \frac{\omega_n}{\omega_0} \tau + \theta_n.$$ 

(23)

Substitution of Eqs. (22) and (23) into Eq. (21) yields

$$\dot{a}_n = g_n^1 \cos \phi_n + g_n^2 \sin \phi_n,$$

$$a_n \dot{\phi}_n = g_n^2 \cos \phi_n - g_n^1 \sin \phi_n,$$

(24)
where

\[
g_1^n = \frac{\gamma_1}{\omega_0} \sin \tau \sum_{m=1}^{N} a_m \left( C_{nm}^{11} \cos \phi_m + C_{nm}^{12} \sin \phi_m \right) \\
+ \gamma_1 \cos \tau \sum_{m=1}^{N} a_m \left( D_{nm}^{11} \cos \phi_m + D_{nm}^{12} \sin \phi_m \right) \\
+ \frac{\kappa}{\omega_0} \sum_{m=1}^{N} a_m \left( E_{nm}^{11} \cos \phi_m + E_{nm}^{12} \sin \phi_m \right) + \frac{\Delta}{\omega} a_n \sin \phi_n
\]

\[
g_2^n = \frac{\gamma_1}{\omega_0} \sin \tau \sum_{m=1}^{N} a_m \left( C_{nm}^{21} \cos \phi_m + C_{nm}^{22} \sin \phi_m \right) \\
+ \gamma_1 \cos \tau \sum_{m=1}^{N} a_m \left( D_{nm}^{21} \cos \phi_m + D_{nm}^{22} \sin \phi_m \right) \\
+ \frac{\kappa}{\omega_0} \sum_{m=1}^{N} a_m \left( E_{nm}^{21} \cos \phi_m + E_{nm}^{22} \sin \phi_m \right) - \frac{\Delta}{\omega} a_n \cos \phi_n.
\]

In Eq. (25), \(C_{nm}^{ij}, D_{nm}^{ij}\) and \(E_{nm}^{ij}\) \((i, j = 1, 2; n, m = 1, 2, \ldots, N)\) respectively denote the elements at \(i\)th row and \(j\)th column of the \(2 \times 2\) submatrices \(C_{mn}, D_{mn}\) and \(E_{mn}\) defined by

\[
C = \begin{pmatrix}
    C_{11} & C_{12} & \cdots & C_{1N} \\
    C_{21} & C_{22} & \cdots & C_{2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    C_{N1} & C_{N2} & \cdots & C_{NN}
\end{pmatrix},
D = \begin{pmatrix}
    D_{11} & D_{12} & \cdots & D_{1N} \\
    D_{21} & D_{22} & \cdots & D_{2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    D_{N1} & D_{N2} & \cdots & D_{NN}
\end{pmatrix},
E = \begin{pmatrix}
    E_{11} & E_{12} & \cdots & E_{1N} \\
    E_{21} & E_{22} & \cdots & E_{2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    E_{N1} & E_{N2} & \cdots & E_{NN}
\end{pmatrix}
\]

4. Subharmonic Resonance

In order to predict the stability of the system implied by Eq. (11), the property of variation of the amplitude \(a_n\) in the long run should be examined. In Eqs. (24), the rate of change of \(a_n\) is small because \(\gamma_1, \kappa\) and \(\Delta\) are all small numbers. For this case, the averaging method is one of the most suitable techniques for estimating the trend of the amplitude variation.

If the speed variation frequency \(\omega\) lies in the neighborhoods of \(2\omega_n\), instability may occur in the form of subharmonic parametric resonance. When the variation frequency approaches \(\omega_0 = 2\omega_n\), an application of the averaging operator\(^{14}\)
results for the case with $\gamma$, $\kappa$ boundaries in the subharmonic resonance are illustrated in Fig. 2, in which $\gamma = 0.0$, 0.002, and 0.005. The effects of the mean axial speed on the stability boundaries in the subharmonic resonance are illustrated in Fig. 1, in which $\beta = 0.8$ and $\kappa = 0.002$ and the solid, dash-dot and dashed lines respectively represent the results for the case with $\gamma = 3.7$, 4.0, and 4.3. The effects of the bending stiffness on the stability boundaries in the subharmonic resonance are illustrated in Fig. 3, in which $\kappa = 0.002$ and $\gamma = 4.0$ and the solid, dash-dot and dashed lines respectively represent the results for the case with $\beta = 0.78$, 0.8 and 0.82.
Fig. 1. Effects of viscosity coefficient on stability boundaries in subharmonic resonance: (a) the first subharmonic resonance; and (b) the second subharmonic resonance.

Fig. 2. Effects of mean axially speed on stability boundaries in subharmonic resonance: (a) the first subharmonic resonance; and (b) the second subharmonic resonance.

Fig. 3. Effects of bending stiffness on stability boundaries in subharmonic resonance: (a) the first subharmonic resonance; and (b) the second subharmonic resonance.
5. Combination Resonance

If the speed variation frequency \( \omega \) lies in the neighborhood of \( \omega_n \pm \omega_m \) \((n \neq m)\), instability may occur in the form of combination parametric resonance. When the variation frequency approaches \( \omega_0 = \omega_n + \omega_m \), the application of the averaging operator to Eqs. (24) yields

\[
\dot{a}_n = (U_{nm}\gamma_1 \cos 2\theta_n + V_n\gamma_1 \sin 2\theta_n + M_n\kappa)a_n \\
\dot{a}_n = (V_n\gamma_1 \cos 2\theta_n - U_{nm}\gamma_1 \sin 2\theta_n + N_n\kappa - \frac{\Delta}{\omega_m})a_n, \tag{32}
\]

where

\[
U_{nm} = \frac{1}{4} \left( D_{nm}^{11} - D_{nm}^{22} + \frac{C_{nm}^{12} + C_{nm}^{21}}{\omega_0} \right) \\
V_{nm} = \frac{1}{4} \left( D_{nm}^{12} + D_{nm}^{21} + \frac{-C_{nm}^{11} + C_{nm}^{22}}{\omega_0} \right) \\
M_n = \frac{E_{nn}^{11} + E_{nn}^{22}}{2\omega_0}, \quad N_n = \frac{E_{nn}^{21} - E_{nn}^{12}}{2\omega_0}.
\]

By expressing in the new variables defined by

\[
\chi_n = a_n \cos \theta_n + ia_n \sin \theta_n \\
\chi_m = a_m \cos \theta_m - ia_m \sin \theta_m,
\]

Eq. (32) takes the form

\[
\dot{\chi}_n = \left( M_n\kappa + iN_n\kappa - i\frac{\Delta}{\omega_n} \right) \chi_n + (U_{nm} + iV_{nm}) \chi_m \\
\dot{\chi}_m = (U_{mn} + iV_{mn}) \chi_n + \left( M_m\kappa + iN_m\kappa - i\frac{\Delta}{\omega_n} \right) \chi_m \tag{35}
\]

for which the characteristic equation

\[
\lambda^2 - \left[ (M_n + M_m + iN_n + iN_m)\kappa + i\Delta \left( \frac{1}{\omega_m} - \frac{1}{\omega_n} \right) \right] \lambda \\
+ \left( M_n\kappa + iN_n\kappa - i\frac{\Delta}{\omega_n} \right) \left( M_m\kappa - iN_m\kappa + i\frac{\Delta}{\omega_m} \right) \\
- (U_{nm} + iV_{nm})(U_{mn} - iV_{mn})\gamma_1^2 = 0. \tag{36}
\]

The analysis of the characteristic roots leads to the stability boundaries in summation parametric resonance at the neighborhood of \( \omega_n + \omega_m \) located in the \((\omega, \gamma_1)\) plane. The contributions of the related parameters to the summation resonance about \( \omega_1 + \omega_2 \) are highlighted in the following figures. The effects of the viscosity coefficient on the stability boundaries in the summation resonance are illustrated in Fig. 4, in which \( \gamma_0 = 4.0 \) and \( \beta = 0.8 \) and the solid, dash-dot and dashed lines respectively represent the results for \( \kappa = 0.0, 0.002 \) and \( 0.005 \). The effects of the mean axial speed on the stability boundaries in the summation resonance are illustrated in Fig. 5, in which \( \beta = 0.8 \) and \( \kappa = 0.002 \) and the solid, dash-dot and dashed lines respectively denote the results for \( \gamma_0 = 3.7, 4.0 \) and \( 4.3 \). The effects of
the bending stiffness on the stability boundaries in the summation resonance are illustrated in Fig. 6, in which \( \kappa = 0.002 \) and \( \gamma_0 = 4.0 \) and the solid, dash-dot and dashed lines represent the results for \( \beta = 0.78, 0.8 \) and 0.82 cases, respectively.

The difference parametric resonance can be analyzed in a similar way, while the numerical results demonstrate that there is no instability region for this case.
6. Note on Two-Term and Four-Term Galerkin Truncations

For given parameters $\gamma_0$ and $\beta$, the exact values of the natural frequencies $\omega_i$ ($i = 1, 2, \ldots$) can be numerically obtained from Eq. (13) via the procedure proposed by Wickert and Mote.\(^{15}\) The two- and four-term Galerkin discretization are used to calculate approximately the first two natural frequencies for varying $\gamma_0$ while $\beta$ is fixed at the value of 0.8. The results are shown in Fig. 7, in which the dashed line and the solid line represent the approximate values obtained from Eq. (11) with $\gamma_1 = 0$ and $\kappa = 0$ for $N = 2$ and $N = 4$, respectively, while the crosses stand for the exact values obtained from Eq. (13). As can be seen, the four-term truncation gives almost the same values for the first and the second natural frequencies for small $\gamma_0$, while the two-term truncation gives an approximate value for the first natural frequency only. For $\gamma_0 = 4.0$ and $\beta = 0.8$, the stability boundaries of the first subharmonic resonance are shown in Fig. 8, in which the crosses and the solid line stand for results based on the two and four-term truncation respectively, and they agree well with each other. It can be inferred that two-term truncation\(^{11}\) is only...

![Figure 7](image1)

Fig. 7. Frequencies of the truncated system compared with the exact values: (a) the first frequency; and (b) the second frequency.

![Figure 8](image2)

Fig. 8. Comparison of two- and four-term truncation.
valid for computing the stability boundaries of the first subharmonic resonance, but not for the second subharmonic case, because the second natural frequency obtained from the two-term truncation is far from the exact value when the axially moving speed is considered a bit higher.

7. Conclusions

The method of averaging is applied to the Galerkin truncation of the governing equation of transverse vibration of an axially accelerating beam. The stability conditions are derived from the averaged equations in the subharmonic and combination resonance. There exist instability regions in the subharmonic resonance and summation resonance, while no instability region exists for the difference resonance. Numerical results show that, in the frequency-amplitude plane, the instability regions move towards the increasing direction of the amplitude with the increase in the viscosity coefficient, and move towards the increasing direction of the frequency as the mean axial speed decreases or as the beam bending stiffness increases. In fact, either the decrease in the mean axial speed or the decrease in the beam bending stiffness leads to increase of the instability threshold of the axial speed fluctuation frequency, which brings about a slight deduction of the instability regions. On the other hand, the increase of the viscosity coefficient leads to a dramatic increase of the instability threshold of the axial speed fluctuation amplitude.

Acknowledgments

The research is supported by the National Natural Science Foundation of China (Project No. 10472060), Natural Science Foundation of Shanghai Municipality (Project No. 04ZR14058) and Shanghai Leading Academic Discipline Project (Project No. Y0103).

References