SOME NOVEL EVOLUTIONAL BEHAVIORS OF LOCALIZED EXCITATIONS IN THE BOITI–LEON–MARTINA–PEMPINELLI SYSTEM

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Using an extended mapping approach and a special Painlevé–Bäcklund transformation, respectively, we obtain two families of exact solutions to the (2+1)-dimensional Boiti–Leon–Martina–Pempinelli (BLMP) system. In terms of the derived exact solution, we reveal some novel evolutional behaviors of localized excitations, i.e., fission, fusion, and annihilation phenomena in the (2+1)-dimensional BLMP system.

Keywords: BLMP system; localized excitation; fission; fusion; annihilation.

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1. Introduction

As is well-known, many dynamical problems in physics and other natural fields are usually characterized by nonlinear evolutional partial differential equations called governing equations. In soliton theory, searing for an analytical exact solution to a nonlinear physical system has long been an important and interesting topic both for physicists and mathematicians, since much physical information and more insight into the physical aspects of a concerned nonlinear problem can be derived from the analytical solution and thus lead to potential applications. In recent decades, much work has been done on this subject, looking for exact solutions and their related properties like the evolutional behaviors of an interaction solution for a nonlinear physical model. For instance, the discovery of dromion type localized coherent structures for the Daver–Stewartson system provided renewed interest in higher-dimensional nonlinear systems.1

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Traditionally, a collision between solitons of integrable models is regarded to be completely elastic.\textsuperscript{2} That is to say, the amplitude, velocity, and wave shape of a soliton do not undergo any change after nonlinear interaction.\textsuperscript{3,4} However, for some special solutions of certain (2+1)-dimensional models in our colleagues’ and our recent study, the interactions among solitonic excitations like peakons and compactons are not completely elastic, since their wave shapes or amplitudes are changed after their collisions.\textsuperscript{5,6} Furthermore, for some (1+1)-dimensional models, two or more solitons may fuse into one soliton at a special time while sometimes one soliton may fission into two or more solitons at other special times.\textsuperscript{7} These phenomena are often called soliton fusion, and soliton fission respectively. Actually, the soliton fusion and fission phenomena have been observed in many physical systems such as organic membrane and macromolecular material,\textsuperscript{8} and physical fields like plasma physics, nuclear physics, and hydrodynamics.\textsuperscript{9} Recently, Wang et al.\textsuperscript{10,11} discussed some (1+1)-dimensional models such as the Burgers equation and the Sharma-Tasso-Olver equation via the Hirota’s direct method, and revealed the solitons fission and soliton fusion phenomena. In a similar way, Zhang\textsuperscript{12} and Lin et al.\textsuperscript{13} also studied the evolutions of soliton solutions for two (1+1)-dimensional nonlinear systems, and found the soliton fission and soliton fusion phenomena again. Now, an important and interesting problem is that there are soliton fission, fusion, and even annihilation phenomena in higher dimensions.\textsuperscript{14} The main purpose of our present paper is to search for some possible localized excitations with fission, fusion, and annihilation behaviors in (2+1)-dimensions. As a concrete example, we consider the following (2+1)-dimensional Boiti–Leon–Martina–Pempinelli system (BLMP)\textsuperscript{15}

\begin{align}
    u_{yt} + (u^2)_{xy} - u_{xxy} + 2v_{xxx} &= 0, \quad (1) \\
    v_t + v_{xx} + 2uv_x &= 0. \quad (2)
\end{align}

The integrability of the above BLMP system was established in Ref. 15. In Ref. 16, it was shown that the BLMP system was Hamiltonian, and it was pointed out that by a certain transformation, the sine-Gordon equation or sinh-Gordon equation can be derived from the BLMP model. These equations arise in some branches of mathematical physics\textsuperscript{17} and have been widely applied in many realistic problems of atomic physics, molecular physics, particle physics, nuclear physics, and shallow water theory.\textsuperscript{18,19} Soliton-like and multisoliton-like solution for this system have been discussed by Lu and Zhang.\textsuperscript{20} However, to the best of our knowledge, studies on its more general solution in some sense, especially certain novel evolutional behaviors such as soliton fission, fusion, and annihilation phenomena for the BLMP system were rarely reported in the preceding literature.

2. Exact Solutions to the (2+1)-Dimensional BLMP System

In this section, we will give out two types of exact solutions, i.e., a mapping solution and a variable separation solution to the BLMP system.
2.1. A mapping solution to the BLMP system

As is known, to search for exact solutions to a nonlinear physical model, one can apply different approaches. One of the most efficient methods of finding soliton excitations of a physical model is the so-called mapping transformation method.\textsuperscript{21-23}

With the help of mapping transformation idea and based on the general reduction theory, we extend the mapping approach. The main idea of the algorithm is that: for a general nonlinear physical system

\[ P(v) \equiv P(x_0 = t, x_1, x_2, \ldots, x_n, v, v_x, v_x, \ldots), \]  

(3)

where

\[ v = v(v_1, v_2, \ldots, v_q)^T, \quad P(v) = (P_1(v), P_2(v), \ldots, P_q(v))^T, \quad P_i(v) \text{ are polynomials of } v_i \text{ and their derivatives} \]  

(\textit{T indicates the transposition of a matrix}). We assume its solution in an extended symmetric form

\[ v_i = \sum_{j=-N}^{N} \alpha_{ij}(x)\phi^j(\omega(x)), \quad x \equiv (t, x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, q, \]  

(4)

where \( \alpha_{ij}(x), \omega(x) \) are arbitrary functions to be determined, \( \phi \) is a solution of the Riccati equation \( \phi' = \sigma + \phi^2 \), \( \sigma \) is a constant and the prime denotes differentiation with respect to \( \omega \). \( N \) is determined by balancing the highest nonlinear terms and the highest-order partial terms in the given nonlinear system.

Substituting the ansatz in Eq. (4) together with the Riccati equation into Eq. (3), collecting coefficients of polynomials of \( \phi \), then setting each coefficient to zero, yields a set of partial differential equations concerning \( \alpha_{ij}(x) \) and \( \omega(x) \). Solving the system of partial differential equations to obtain \( \alpha_{ij}(x) \) and \( \omega(x) \), substituting the derived results and the solutions of Riccati equation into Eq. (4), one can derive exact solutions to the given nonlinear system.

According to the above idea and by the balancing procedure, the ansatz in Eq. (4) becomes

\[ u = f + g\phi(q) + h\phi^{-1}(q), \quad v = \alpha + \beta\phi(q) + \gamma\phi^{-1}(q), \]  

(5)

where \( f, g, h, \alpha, \beta, \gamma, \) and \( q \) are arbitrary functions of \( \{x, y, t\} \) to be determined. Substituting Eq. (5) with the Riccati equation into Eqs. (1) and (2), and collecting coefficients of polynomials of \( \phi \), then eliminating each coefficient, yields a set of partial differential equations of \( f, g, h, \alpha, \beta, \gamma, \) and \( q \). Taking careful calculation with the aid of Mathematica or Maple, one can derive an exact solution as follows

\[ \alpha = \int \frac{q_{xx}q_{xy} + q_t q_{xy} - q_{x}q_{xy}q_x - q_{xy}q_{yt}}{2q_x^2} \, dx, \quad \beta = -q_y, \quad \gamma = \sigma q_y, \]  

\[ f = -\frac{q_t + q_{xx}}{2q_x}, \quad g = -q_x, \quad h = \sigma q_x, \]  

(6)

with

\[ q = \chi(x, t) + \varphi(y), \]  

(7)
where $\chi \equiv \chi(x, t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable-separated functions of $(x, t)$ and $y$, respectively.

Finally, based on the solution of Riccati equation $\phi' = \sigma + \phi^2$

$$\phi = \begin{cases} -\sqrt{-\sigma} \tanh(\sqrt{-\sigma} \omega), & \sigma < 0, \\ \sqrt{\sigma} \tan(\sqrt{\sigma} \omega), & \sigma > 0, \\ -\frac{1}{\omega}, & \sigma = 0, \end{cases}$$

(here we have omitted the coth-type and cot-type solutions due to their concomitance property) substituting the derived result (6) and the solution (8) into the above ansatz (5), one can obtain exact mapping solutions to the BLMP system.

**Case 1**

For $\sigma < 0$, we can derive the following solitary wave solutions to the BLMP system

$$u_1 = -\frac{2\sigma \chi_x^2 \tanh^2(\sqrt{-\sigma}(\chi + \varphi)) + \sqrt{-\sigma} \tanh(\sqrt{-\sigma}(\chi + \varphi)) (\chi_{xx} + \chi_t) + 2\sigma \chi_x^2}{2\chi_x \sqrt{-\sigma} \tanh(\sqrt{-\sigma}(\chi + \varphi))},$$

$$v_1 = -\frac{\sigma \varphi_y [\tanh^2(\sqrt{-\sigma}(\chi + \varphi)) + 1]}{\sqrt{-\sigma} \tanh(\sqrt{-\sigma}(\chi + \varphi))},$$

with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$.

**Case 2**

For $\sigma > 0$, we can obtain the following periodic wave solutions to the BLMP system

$$u_2 = -\frac{2\sqrt{\sigma} \chi_x^2 \tanh^2(\sqrt{\sigma}(\chi + \varphi)) + \tan(\sqrt{\sigma}(\chi + \varphi)) (\chi_{xx} + \chi_t) - 2\sqrt{\sigma} \chi_x^2}{2\chi_x \tan(\sqrt{\sigma}(\chi + \varphi))},$$

$$v_2 = -\frac{\sqrt{\sigma} \varphi_y [\tanh^2(\sqrt{\sigma}(\chi + \varphi)) - 1]}{\tan(\sqrt{\sigma}(\chi + \varphi))},$$

with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$.

**Case 3**

For $\sigma = 0$, we can derive the following variable separation solution to the BLMP system

$$u_3 = -\frac{\chi_t + \chi_{xx}}{2\chi_x} + \frac{\chi_x}{\chi + \varphi},$$

$$v_3 = \frac{\varphi_y}{\chi + \varphi},$$

also with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$.
2.2. A variable separation solution to the BLMP system

Now, we further search for a more general exact solution based on the multilinear variable separation approach (MLVSA) for the BLMP system. The key idea of this method is to make the original field such as \( u \) be a function of several reduced fields, say, two reduced fields, \( P \) and \( Q \),

\[
  u(x, y, t) = \varphi(x, y, t, P(x, t), Q(y, t)),
\]

where \( P \equiv P(x, t) \) is \( y \)-independent and \( Q \equiv Q(y, t) \) is \( x \)-independent. This method has been revisited and successfully applied to various (2+1)-dimensional models like the (2+1)-dimensional AKNS system, the nonlinear Schrödinger equation (NLS), the Nizhnik–Novikov–Veselov (NNV) model, and the higher-order Broer–Kaup equations (HBK), etc.\(^3\)\(^{-6}\) As a matter of fact, a nearly systematic process of the MLVSA to solve (2+1)-dimensional soliton systems has been accomplished. Nevertheless, the MLVSA is still in progress, aiming at deriving more general excitations in the sense that it admits more arbitrary separation functions entering into the solutions.

First, we take the following Painlevé–Bäcklund transformation to \( u \) and \( v \) in Eqs. (1) and (2)

\[
  u = \sum_{j=0}^{\alpha_1} u_j f^{j-\alpha_1}, \quad v = \sum_{j=0}^{\alpha_2} v_j f^{j-\alpha_2},
\]

where \( u_{\alpha_1} \) and \( v_{\alpha_2} \) are arbitrary seed solutions of the BLMP system. By using the leading term analysis, we obtain \( \alpha_1 = \alpha_2 = 1 \). Substituting Eq. (16) into Eqs. (1) and (2), and considering the fact that the functions \( u_1 \) and \( v_1 \) are seed solutions of the original model, reads

\[
  \sum_{i=0}^{3} \psi_{1i} f^{i-4} = 0, \quad \sum_{i=0}^{2} \psi_{2i} f^{i-3} = 0,
\]

where \( \psi_{1i}, \psi_{2i} \) are the functions of \( \{u_j, v_j, f\} \) and their derivatives. Thanks to the complexity of the expression of \( \psi_{1i} \) and \( \psi_{2i} \), we neglect their concrete forms. Eliminating the leading terms of Eq. (17), the functions \( \{u_0, v_0\} \) are determined. Inserting the results into Eq. (16) and rewriting its form, the Painlevé–Bäcklund transformation becomes

\[
  u = (\ln f)_x + u_1, \quad v = (\ln f)_y + v_1.
\]

For convenience of discussion, we choose the seed solutions \( u_1 \) and \( v_1 \) to be \( u_1 = u_1(x, t), \quad v_1 = 0 \), where \( u_1(x, t) \) is an arbitrary function of indicated arguments. Substituting Eq. (18) together with the seed solution into Eqs. (1) and (2) yields

\[
  [f^2 \partial_{xy} - f(f_x \partial_y + f_y \partial_x + f_{xy}) + 2 f_x f_y (f_t + f_{xx} + 2 u_1 f_x)] = 0, \quad (19)
\]

\[
  [f \partial_y - f_y] (f_t + f_{xx} + 2 u_1 f_x) = 0. \quad (20)
\]
Obviously, Eqs. (19) and (20) can be reduced to a linear equation
\[ f_t + f_{xx} + 2u_1 f_x = 0. \] (21)

Since Eq. (21) is a linear equation, one can certainly make use of the linear superposition theorem. For instance
\[ f = Q_0(y) + \sum_{k=1}^{N} P_k(x,t)Q_k(y,t), \] (22)
where \( P_k(x,t) \equiv P_k \) and \( Q_k(y,t) \equiv Q_k \) are simply the functions of \( \{x,t\} \) and \( \{y,t\} \), respectively, and \( Q_0(y) \equiv Q_0 \). Inserting Eq. (22) into Eq. (21) yields the following variable separation equations
\[ P_{kt} + P_{kxx} + 2u_1 P_{kx} + \Gamma_k(t)P_k = 0, \quad Q_{kt} - \Gamma_k(t)Q_k = 0, \quad (k = 1, 2, \ldots, N), \] (23)
where \( \Gamma_k(t) \) is an arbitrary function of time \( t \). Then, a general variable separation solution to the BLMP system yields
\[ u = \frac{\sum_{k=1}^{N} P_{kxx}Q_k}{Q_0 + \sum_{k=1}^{N} P_k Q_k} + u_1, \quad v = \frac{\sum_{k=1}^{N} P_k Q_k}{Q_0 + \sum_{k=1}^{N} P_k Q_k}, \] (24)
where \( u_1, P_k, \) and \( Q_k \) admit Eq. (23).

The corresponding general potential \( G (G \equiv u_y \equiv v_x) \) of the BLMP system reads
\[ G = \frac{\sum_{k=1}^{N} P_{kxx}Q_k}{Q_0 + \sum_{k=1}^{N} P_k Q_k} - \frac{\sum_{k=1}^{N} P_{kxx}Q_k(Q_0 + \sum_{k=1}^{N} P_k Q_k)}{(Q_0 + \sum_{k=1}^{N} P_k Q_k)^2}. \] (25)

In order to discuss some interesting properties of the general variable separation solution (24) or the general potential \( G \), we need to make further simplifications. Based on Eqs. (24) and (25), if setting \( N = 3, \Gamma_k(t) = 0, Q_0 = a_0, Q_1 = a_1, Q_2 = a_2 Q, Q_3 = a_3 Q, (a_i = \text{consts}, i = 0, \ldots, 3), P_1 = P_3 = P \) and \( P_2 = 1 \), then Eqs. (22) and (23) become
\[ f = a_0 + a_1 P + a_2 Q + a_3 P Q, \] (26)
\[ P_t - P_{xx} - 2u_1 P_x = 0, \quad Q_t = 0. \] (27)
Finally, we can derive a special variable separation solution and a special potential for the BLMP system
\[ u = \frac{(a_1 + a_3 Q)P_x}{a_0 + a_1 P + a_2 Q + a_3 P Q} - \frac{P_t + P_{xx}}{2P_x}, \] (28)
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\[ v = \frac{(a_2 + a_3 P)Q_y}{a_0 + a_1 P + a_2 Q + a_3 PQ}, \quad (29) \]

\[ G = \frac{(a_3 a_0 - a_2 a_1)P_x Q_y}{(a_0 + a_1 P + a_2 Q + a_3 PQ)^2}, \quad (30) \]

with two arbitrary functions \( P(x,t) \equiv P, \ Q(y) \equiv Q \).


Comparing the special variable separation solutions \( u \) and \( v \) expressed by Eqs. (28) and (29) with the mapping solutions shown by Eqs. (13) and (14) in Case 3, one can find that they are evidently equivalent via a simple variable transformation \((\chi = (a_0 + a_1 P)/(a_2 + a_3 P), \varphi = Q)\). Therefore, the mapping solutions are special cases of the general variable separation solution. Additionally, by contrast, the special potential \( G \) (30) with the so-called common formula (1.1) in Ref. 3, one may be surprised to find that they are completely identical via a simple scale transformation: \( U = -2G \). Therefore, all the localized excitations based on the common formula (1.1), such as dromions, lumps, breathers, instantons, fractal and chaotic patterns obtained in Ref. 3, can be redevised in the BLMP system. Since these localized structures have been widely reported in the previous literature,\textsuperscript{24-28} we neglect the related discussions in the section. Actually, based on the special solitary wave solutions in Case 1 and the periodic wave solutions in Case 2, one can also find abundant stable localized excitations similar to the cases reported in the proceedings literature.\textsuperscript{29,30}

In this section, we do not study the general potential \( G \) expressed by Eq. (25), and only discuss the special potential \( G \) shown by Eq. (30) owing to its universal validity properties in \((2+1)\)-dimensions. Virtually even in this simple situation, one can still find some localized structures with novel evolutional properties for the \((2+1)\)-dimensional nonlinear system when prescribing the functions \( P \) and \( Q \) appropriately to avoid some singularities. In other words, the special potential \( G \) (30) may possess some novel properties that have not been revealed.

Recently, it has been reported theoretically that the fission and fusion phenomena can happen for \((1+1)\)-dimensional solitons or solitary waves.\textsuperscript{10,11,14} Now, we focus our attention on these intriguing fusion, fission, and annihilation phenomena for the special potential \( G \) (30) in \((2+1)\)-dimensions, which may exist in certain situations. For instance, when we select the arbitrary functions \( P(x,t) \) and \( Q(y) \) in Eq. (30) to be

\[ P = 1 + \sum_{i=1}^{M} C_i \exp[K_i(x + \omega_i t)] + \sum_{i=1}^{N} \begin{cases} X_i(x + v_i t), & \text{if } x + v_i t \leq 0, \\ -X_i(-x - v_i t) + 2X_i(0), & \text{if } x + v_i t > 0, \end{cases} \quad (31) \]
where $M, N,$ and $L$ are positive integers, $X_i(x+v_i t) = X_i(x)$ is a differential function of the indicated argument, then we can obtain a new type of fission solitary wave solution for the potential $G$ (30). Figure 1 shows an evolutional profile of the solitary wave solution for the potential $G$ with conditions: $X_1 = 0.1 \exp(x + t)$, $M = N = L = K_1 = S_1 = C_1/5 = 10D_1 = -\omega_1/2 = a_0 = a_3 = 1$, $a_1 = a_2 = y_0 = 0$. From Figs. 1(a)–1(d), one can clearly see one soliton fissions into two solitons. It is interesting to mention that the left travelling soliton [see Fig. 1(e)], i.e., one of the pairs of solitons that emerge after the fission, is stable and do not undergo additional fissions as the running program for longer period of time until $t = 10^6$. However,

$$Q = 1 + \sum_{j=1}^{L} D_j \tanh[S_j(y + y_0)], \quad (32)$$

**Fig. 1.** The evolutional profile of one soliton fission into two solitons for the potential $G$ (30) under conditions (31) and (32) with the following parameters $X_1 = 0.1 \exp(x + t)$, $M = N = L = K_1 = S_1 = C_1/5 = 10D_1 = -\omega_1/2 = a_0 = a_3 = 1$, $a_1 = a_2 = y_0 = 0$ at different times: (a) $t = -5$, (b) $t = 1$, (c) $t = 5$, (d) $t = 10$. (e) The stable left travelling soliton at $t = 15$. (f) The unstable right travelling soliton at $t = 15$. 
the right travelling soliton is unstable [see Fig. 1(f)] and will fission further into many oscillating solitons as time \( t > 15 \), their shapes and amplitudes are changed with times.

Along with the above line, when we consider \( P(x, t) \) and \( Q(y) \) to be

\[
P = 1 + 0.3 \tanh[5(x - t)] + 0.25 \tanh(2x - 3t) + 0.1 \tanh[2(x - 2t)] \over [1 + 0.1 \tanh(2x - 3t)]^2, \tag{33}
\]

\[
Q = 1 + 0.1 \tanh(y), \tag{34}
\]

and \( a_0 = a_3 = 1, a_1 = a_2 = 0 \) for the potential \( G(30) \), then we obtain another new type of fusion solitary wave, which possesses an apparently different property compared with Fig. 1. From Fig. 2, one can find that three solitons fuse into one soliton finally. The fused single soliton remains stable for subsequent times as the running program for a rather long time \( (t = 10^6) \).

In addition to the fission and fusion of the above-mentioned localized excitations with the increase in time, some solitons may be annihilated in some appropriate initial and/or boundary conditions like other physical particles. For instance, when prescribing \( P(x, t) \) and \( Q(y) \)

\[
P = 1 + 0.3 \tanh(x^2 + t), \quad Q = 1 + 0.1 \tanh(y^2), \tag{35}
\]

and \( a_0 = a_3 = 1, a_1 = a_2 = 0 \) for the potential \( G(30) \), then we find an interesting annihilation phenomenon of solitons for the potential \( G(30) \). The corresponding profile is presented in Fig. 3.

![Graphs](image-url)
Naturally, if we consider another selection for the functions $P$ and $Q$ such as Jacobian functions or the solutions of the well-known Lorenz chaotic system, then we may derive some novel solitary wave solutions with double periodic properties or chaotic behaviors, which are omitted in our present paper. Actually, because there exist some arbitrary characteristic functions $P$ and $Q$ in the potential $G$, any exotic behaviors may engender along with the above-mentioned lines.

4. Summary and Discussion

In summary, applying an extended mapping approach and a special Painlevé–Bäcklund transformation, respectively, two families of exact solutions to the (2+1)-
dimensional BLMP system are successfully obtained. Based on the derived exact solution with arbitrary functions, we list three novel examples, soliton fission, fusion, and annihilation phenomena for the BLMP system. Here, one may ask: are there indications in nature for nonelastic interactions among solitons? The answer is positive. Conventionally, it is usually considered that the interactions among solitons are completely elastic due to their amplitudes, velocities, and wave shapes of solitons completely preserving after nonlinear interaction. However, for the localized solutions obtained here, it occurred that their interactions lead to new physical properties like fusion, fission, and annihilation of these solutions. In these special cases, the soliton collisions are nonelastic or completely nonelastic, since their amplitudes and wave shapes of solitons are changed or eliminated after their nonlinear interactions.

From the brief analysis in our present paper, we can see that the soliton fission, fusion, and annihilation intriguing phenomena can occur in a higher-dimensional soliton system if we choose appropriate initial conditions or boundary conditions, which are similar to some work in (1+1)-dimensions.\cite{10-13} Actually, the soliton fusion, fission, and annihilation phenomena have been observed in some physical systems such as organic membrane and macromolecular material, and some physical fields like nuclear physics, plasma physics, and hydrodynamics. Although we have given out some soliton fusion and fission phenomena in (2+1)-dimensions, it is obvious that there are still many significant and interesting problems waiting for further discussion. Just as the authors\cite{10,11} have pointed out in (1+1)-dimensional cases, what is the necessary and sufficient condition for soliton fission and soliton fusion? What is the general equation for the distribution of the energy and momentum after soliton fission and soliton fusion? How can we use the soliton fission and soliton fusion of integrable models to practically investigate observed soliton fission and soliton fusion in the experiments? These are all pending issues. Actually, our present short note is an initial work, due to widely potential applications of soliton theory, to learn more about the soliton fission, fusion, and annihilation properties and their applications in reality, which are worthy of further study.

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