Dynamic stability of an axially accelerating viscoelastic beam

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Abstract

This work investigates dynamic stability in transverse parametric vibration of an axially accelerating viscoelastic tensioned beam. The material of the beam is described by the Kelvin model. The axial speed is characterized as a simple harmonic variation about the constant mean speed. The Galerkin method is applied to discretize the governing equation into a infinite set of ordinary-differential equations under the fixed–fixed boundary conditions. The method of averaging is employ to analyze the dynamic stability of the 2-term truncated system. The stability conditions are presented and confirmed by numerical simulations in the case of subharmonic and combination resonance. Numerical examples demonstrate the effects of the dynamic viscosity, the mean axial speed and the tension on the stability conditions.

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1. Introduction

Axially moving beams can represent many engineering devices, such as band saws and serpentine belts. Despite many advantages of these devices, vibrations associated with the devices have limited their applications. One major problem is the occurrence of large transverse vibrations due to tension or axial speed variation termed as parametric vibrations.

Axial transport acceleration frequently appears in engineering systems. For example, if an axially moving beam models a belt on a pair of rotating pulleys, the rotation vibration of the pulleys will result in a small fluctuation in the axial speed of the belt. Although vibration analysis of parametrically excited, axially moving beams has been studied extensively, the literature that is specially related to the axially accelerating beams is relatively limited. Pasin (1972) studied the stability of transverse vibrations of beams with periodically reciprocating motion in axial direction. Öz, Pakdemirli and Özkaya (1998) applied the method of multiple scales to study dynamic stability of an axially accelerating beam with small bending stiffness. Özkaya and Pakdemirli (2000) applied the method of multiple scales and the method of matched asymptotic expansions to construct non-resonant boundary layer solutions for an axially accelerating beam with small bending stiffness. Öz, Pakdemirli (2000) and Öz (2001) used the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating tensioned beam under simply-supported conditions and fixed–fixed conditions respectively. Parker and Lin (2001) adopted a 1-term Galerkin discretization and the perturbation method to study dynamic stability of an axially accelerating beam subjected to a tension fluctuation. Özkaya and Öz (2002) applied artificial neural network algorithm to determine stability of an axially accelerating beam.

All above-mentioned researchers assumed the beam under their consideration is elastic, and did not account for any dissipative mechanisms. However, the modeling of dissipative mechanisms is an important research topic of axially
moving material vibrations (Wickert and Mote, 1988; Abrate, 1992). Viscoelasticity is an effective approach to model the dissipative mechanism because some beam-like engineering devices are composed of some viscoelastic metallic or ceramic reinforcement materials like glass-cord and viscoelastic polymeric materials such as rubber. Based on 3-term Galerkin truncation, Marynowski (2002) and Marynowski and Kapitaniak (2002) compared the Kelvin model with the Maxwell model and the Bügers model respectively through numerical simulation of nonlinear vibration responses of an axially moving beam at a constant speed. They found that in the case of small damping, all models yield similar results. To authors’ knowledge, there are no researches on transverse vibration of an axially accelerating viscoelastic beam. To address the lack of research in this aspect, this paper investigates dynamic stability of an axially accelerating viscoelastic beam.

2. The governing equation and its Galerkin discretization

A uniform axially moving viscoelastic beam, with linear density \( \rho \), cross-sectional area moment of inertia \( I \) and initial tension \( P \), travels at the time-dependent axial transport speed \( c(T) \) between two prismatic ends separated by distance \( L \). Because only small damping is considered here, according to the results of Marynowski (2002) and Marynowski and Kapitaniak (2002), one can assume the viscoelasticity of the beam material to be defined by the Kelvin model

\[
\sigma = E_0 \left( 1 + \alpha \frac{\partial}{\partial T} \right) \varepsilon,
\]

where \( \sigma \) is the axial stress, \( \varepsilon \) is the axial strain, \( E_0 \) is the stiffness constant, and \( E_0 \alpha \) is the viscosity coefficient. Consider only the bending vibration described by the transverse displacement \( V(X, T) \), where \( T \) is the time, and \( X \) is the axial coordinate.

The Newton second law of motion yields

\[
\rho \frac{\partial^2 V}{\partial t^2} + 2\frac{\partial^2 V}{\partial X \partial T} + \frac{d c}{dT} \frac{\partial V}{\partial X} + \epsilon \frac{\partial^2 V}{\partial X^2} - P \frac{\partial^2 V}{\partial X^2} - E_0 \left( 1 + \alpha \frac{\partial}{\partial T} \right) \frac{\partial^4 V}{\partial X^4} = 0.
\]

(2)

The prismatic joints between which the beam travels can be modeled as fixed-fixed ends. Hence the boundary conditions are

\[
V(0, t) = V(L, t) = 0, \quad \frac{\partial V}{\partial X}(0, t) = \frac{\partial V}{\partial X}(L, t) = 0.
\]

(3)

It is assumed that the transport speed is characterized as a simple harmonic variation about the constant mean speed, i.e.,

\[
c(T) = c_0 (1 + h \cos \Omega T).
\]

(4)

The assumption has its the physical meaning. For example, if the axially moving beam models a belt on a pair of rotating pulleys, the rotation vibration of the pulleys will result in a small fluctuation in the axial speed of the belt. Substituting Eq. (4) into Eq. (2) and transforming the resulting equation into the dimensionless form yield the dimensionless governing equation of transverse motion

\[
\frac{\partial^2 v}{\partial t^2} + 2\gamma (1 + h \cos \omega t) \frac{\partial^2 v}{\partial X \partial t} + \gamma h \cos \omega t \frac{\partial^2 v}{\partial x^2} + (\gamma^2 - \mu^2 + 2\gamma \mu \cos \omega t) \frac{\partial^2 v}{\partial x^4} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^4} + \frac{\partial^2 v}{\partial x^6} = 0,
\]

(5)

where

\[
\frac{v}{L}, \quad \frac{x}{L}, \quad \frac{t}{T}, \quad \frac{X}{L}, \quad \frac{E_0 I}{\rho L^4}, \quad \gamma = \frac{c_0}{\sqrt{\frac{\rho L^2}{E_0 I}}}, \quad \omega = \frac{\Omega L^2}{\rho L^4}.
\]

(6)

The corresponding dimensionless boundary conditions are

\[
\nu(0, t) = \nu(1, t) = 0, \quad \frac{\partial \nu}{\partial x}(0, t) = \frac{\partial \nu}{\partial x}(1, t) = 0.
\]

(7)

The Galerkin method is employed to simplify Eq. (5). Under given boundary conditions, the solution of Eq. (5) may be expanded into the series of eigenfunctions of the stationary beam with two fixed ends

\[
\nu(x, t) = \mathbf{q}(t)^T \mathbf{\varphi}(x),
\]

(8)

where the infinite column matrices \( \mathbf{q}(t) \) and \( \mathbf{\varphi}(x) \) are respectively assembled by the generalized displacements and the stationary eigenfunctions, namely

\[
\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_n(t), \ldots)^T, \quad \mathbf{\varphi}(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots)^T.
\]

(9)
For the beam under the fixed–fixed boundary conditions (7), the stationary eigenfunctions are (Bishop and Johnson, 1979)
\[
\varphi_i(x) = \cos \beta_i x - \cosh \beta_i x - \cos \beta_i - \cosh \beta_i (\sin \beta_i x - \sinh \beta_i x),
\]
where \( \beta_i \) is the \( i \)th root of the frequency equation
\[
\cos \beta_i \cosh \beta_i - 1 = 0.
\]
Taking the appropriate derivatives and substituting them into Eq. (5), one obtains the residual
\[
R(x, t) = \ddot{\varphi}^T \varphi + [2\gamma (1 + h \cos \omega t) \ddot{q}^T - \gamma \omega h \sin \omega t \dot{q}^T] \varphi' + (\gamma^2 - \mu^2 + 2\gamma^2 h \cos \omega t) q^T \varphi'' + (q^T + \eta \dot{q}^T) \varphi'''.
\]
If the weighting functions are also chosen as the stationary beam eigenfunctions, then application of the Galerkin method requires that the residual (12) should satisfy
\[
\int_0^1 R(x, t) \varphi(x) \, dx = 0\bigg|_\infty,
\]
where \( 0\bigg|_\infty \) is an infinite zero column matrix. Substituting Eq. (12) into Eq. (13) and transposing the resulting equation, one obtains the Galerkin discretization of the governing equation
\[
I_\infty \ddot{\varphi}^T + B(2\gamma (1 + h \cos \omega t) \ddot{q}^T - \gamma \omega h \sin \omega t \dot{q}^T) \varphi' + (\gamma^2 - \mu^2 + 2\gamma^2 h \cos \omega t) Cq^T + A(q^T + \eta \dot{q}^T) = 0\bigg|_\infty,
\]
where
\[
I_\infty = \int_0^1 \varphi(x) \varphi(x)^T \, dx, \quad B = \int_0^1 \varphi(x) \varphi'(x)^T \, dx,
\]
\[
C = \int_0^1 \varphi(x) \varphi''(x)^T \, dx, \quad A = \int_0^1 \varphi(x) \varphi'''(x)^T \, dx.
\]
The orthonormality of the stationary beam eigenfunctions leads to
\[
I_\infty = \text{diag}[1 \quad 1 \quad \ldots], \quad A = \text{diag}[\beta_1^4 \quad \beta_2^4 \quad \ldots \quad \beta_n^4 \quad \ldots].
\]
In the case that \( h = 0 \) and \( \eta = 0 \), Eq. (5) is reduced to the generating system
\[
\frac{\ddot{u}}{\partial t^2} + 2\gamma^2 v + (\gamma^2 - \mu^2) \frac{\ddot{u}}{\partial x^2} + \frac{\ddot{u}}{\partial x^4} = 0,
\]
which governs free transverse vibration of an elastic beam axially moving at a constant speed. Following the procedure proposed by Wickert and Mote (1990), one can determine numerically the natural frequencies \( \omega_i \) (\( i = 1, 2, \ldots \)) for given parameters \( \gamma \) and \( \mu \).

3. The averaged equation and stability analysis

In present investigation, the authors will consider subharmonic resonance and combination resonance near the first two eigenfrequencies of the generating system (17). Therefore only first two terms are retained in the Galerkin expansion (8). In this case, both \( B \) and \( C \) are \( 2 \times 2 \) matrices, and direct calculations yield
\[
B = b J_2, \quad C = \text{diag}[c_1 \quad c_2],
\]
where
\[
b = \frac{4\beta_1^2 \beta_2^2}{\beta_1^4 - \beta_2^4} \left[ 1 + \frac{(\tan \beta_1 - \tanh \beta_1)(\tan \beta_2 - \tanh \beta_2)}{(\sin \beta_1 - \sinh \beta_1)(\sin \beta_2 - \sinh \beta_2)} \right], \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
\[
c_i = -\beta_i \left[ 2 \cosh \beta_i \sin \beta_i + (\cosh 2 \beta_i - 3) \sin 2 \beta_i + 4(\cos \beta_i - \cosh \beta_i)^2 \\ + (\cosh 2 \beta_i + 4 \cos \beta_i - 3) \sinh 2 \beta_i / [4(\sin \beta_i - \sinh \beta_i)^2] \right] (i = 1, 2).
\]
Introduce a new set of state variables
\[
\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^T = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T.
\]
Substitution of Eqs. (29) and (30) into Eq. (27) leads to

$$\dot{\xi} = A\xi + hA_1\xi \cos \omega t + \omega hA_2\xi \sin \omega t + \eta A_3\xi.$$  

(21)

where

$$A = -\begin{pmatrix} 0_2 & 0_2 \\ (\gamma^2 - \mu^2)C + A & I_2 \end{pmatrix}, \quad A_1 = -2\gamma \begin{pmatrix} 0_2 & 0_2 \\ -\gamma C & B \end{pmatrix}, \quad A_2 = \gamma \begin{pmatrix} 0_2 & 0_2 \\ B & 0_2 \end{pmatrix},$$

$$A_3 = -n \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & A \end{pmatrix}, \quad 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

(22)

The matrix $A$ describes the dynamics of transverse vibration of an elastic beam axially moving at a constant speed. When the speed is lower than the critical speed, matrix $A$ has four pure imaginary eigenvalues, $\pm i\omega_1$ and $\pm i\omega_2$. Thus there exists canonical transformation matrix $T$ such that

$$T^T AT = \Omega = \begin{pmatrix} -\omega_1 J_2 & 0_2 \\ 0_2 & -\omega_2 J_2 \end{pmatrix}.$$  

(23)

Introduce the state variable transformation

$$\xi = T\zeta.$$  

(24)

Then Eq. (21) can be expressed in terms of new state variables as

$$\dot{\xi} = \Omega \xi + hD_1\xi \cos \omega t + \omega hD_2\xi \sin \omega t + \eta D_3\xi.$$  

(25)

where

$$D_1 = T^T A_1 T, \quad D_2 = T^T A_2 T, \quad TD_3 = T^T A_3 T.$$  

(26)

If the speed variation frequency $\omega$ approaches $\omega_0 = 2\omega_1$, $2\omega_2$, or $\omega_1 \pm \omega_2$, instability may occur in subharmonic or combination parametric resonance. A small parameter $\delta$ is introduced to quantify the deviation of $\omega$ from $\omega_0$, and $\omega$ is described by $\omega = \omega_0(1 + \delta)$. Introduce a new time $\tau = \omega t$. Eq. (25) can be rewritten as

$$\dot{\xi} = \Omega_0 \xi - \delta \Omega_0 \xi + hD_4\xi \cos \tau + \omega D_5\xi \sin \tau + \eta D_6\xi,$$  

(27)

where

$$\Omega_0 = \frac{\Omega}{\omega_0} = \begin{pmatrix} -k_1 J_2 & 0_2 \\ 0_2 & -k_2 J_2 \end{pmatrix}, \quad k_1 = \frac{\omega_1}{\omega_0}, \quad k_2 = \frac{\omega_2}{\omega_0}.$$  

(28)

In Eq. (27), $\delta$, $h$ and $\eta$ are all small parameters.

To apply the averaging method, rewrite $\xi$ in an amplitude-phase form

$$\xi_1 = a_1 \cos \varphi_1, \quad \xi_2 = a_1 \sin \varphi_1, \quad \xi_3 = a_2 \cos \varphi_2, \quad \xi_4 = a_2 \sin \varphi_2,$$  

(29)

where

$$\varphi_j = k_j \tau + \theta_j \quad (j = 1, 2).$$  

(30)

Substitution of Eqs. (29) and (30) into Eq. (27) leads to

$$\ddot{a}_1 \cos \varphi_1 + g_1 \sin \varphi_1, \quad a_1 \dot{\varphi}_1 = g_2 \cos \varphi_1 - g_1 \sin \varphi_1,$$

$$\dot{a}_2 \cos \varphi_2 + g_3 \sin \varphi_2, \quad a_2 \dot{\varphi}_2 = g_4 \cos \varphi_2 - g_3 \sin \varphi_2,$$  

(31)

where

$$g_1 = (D_4^{1,i}a_1 \cos \varphi_1 + D_4^{2,i}a_1 \sin \varphi_1 + D_4^{3,i}a_2 \cos \varphi_2 + D_4^{4,i}a_2 \sin \varphi_2)h/\omega_0 \cos \tau$$

$$+ (D_2^{2,i}a_1 \cos \varphi_1 + D_2^{3,i}a_1 \sin \varphi_1 + D_2^{4,i}a_2 \cos \varphi_2 + D_2^{4,i}a_2 \sin \varphi_2)h \sin \tau$$

$$+ (D_3^{3,i}a_1 \cos \varphi_1 + D_3^{4,i}a_1 \sin \varphi_1 + D_3^{4,i}a_2 \cos \varphi_2 + D_3^{4,i}a_2 \sin \varphi_2)\eta/\omega_0 + \delta r_1.$$  

(32)

In Eq. (32), the superscript of an element denotes its row and column in the matrix, and

$$r_1 = k_1 a_1 \sin \varphi_1, \quad r_2 = -k_1 a_1 \cos \varphi_1, \quad r_3 = k_2 a_2 \sin \varphi_2, \quad r_4 = -k_2 a_2 \cos \varphi_2.$$  

(33)
For small parameters $\delta$, $h$ and $\eta$, both the amplitudes and the phases are slowly varying with time. Therefore, the right-hand side of Eq. (31) can be replaced by their time averages over a period, and the solutions of the averaged system are the same as the original one (Bogoliubov and Mitropolsky, 1961). Ariaratnam and Namchchivaya (1986) and Asokanthan and Ariaratnam (1994) applied the method of averaging to study axially moving materials. The averaged equations take different forms depending on the value of the speed variation frequency.

For $k_j = 1/2$ ($j = 1, 2$), the variation frequency lies in the neighborhoods of $2\omega_j$. The non-trivial components of the averaged equation take the form

$$a_j = (U_j h \cos 2\theta_j + V_j h \sin 2\theta_j + M_j \eta) a_j,$$

$$a_j \dot{\theta}_j = (V_j h \cos 2\theta_j - U_j h \sin 2\theta_j + N_j \eta - \frac{1}{2} \delta) a_j \quad (j = 1, 2),$$

where

$$U_1 = \frac{1}{4} \left( \frac{D_{1,1}^{1.1} - D_{1,2}^{1.2}}{\omega_0} + D_{2,2}^{1.2} + D_{2,1}^{2.1} \right), \quad V_1 = \frac{1}{4} \left( \frac{D_{1,1}^{1.2} + D_{1,2}^{2.1}}{\omega_0} - D_{2,1}^{1.1} + D_{2,2}^{2.2} \right),$$

$$M_1 = \frac{D_{3,1}^{1.1} + D_{3,2}^{2.2}}{2\omega_0}, \quad N_1 = \frac{D_{3,1}^{1.2} - D_{3,2}^{2.1}}{2\omega_0}, \quad U_2 = \frac{1}{4} \left( \frac{D_{1,3}^{3,3} - D_{1,4}^{4,4}}{\omega_0} + D_{2,2}^{3,4} + D_{2,3}^{4,3} \right),$$

$$V_2 = \frac{1}{4} \left( \frac{D_{1,4}^{4,4} + D_{1,3}^{3,3}}{\omega_0} - D_{2,2}^{3,3} + D_{2,4}^{4,4} \right), \quad M_2 = \frac{D_{3,3}^{3,3} + D_{3,4}^{4,4}}{2\omega_0}, \quad N_2 = \frac{D_{3,3}^{3,3} - D_{3,4}^{4,4}}{2\omega_0}.$$

Introducing two new variables

$$\chi_j = a_j \cos(\theta_j + \varphi_j), \quad \varsigma_j = a_j \sin(\theta_j + \varphi_j) \quad (j = 1, 2),$$

where

$$\varphi_j = \frac{1}{2} \tan^{-1} \left( \frac{U_j}{V_j} \right).$$

Then Eq. (34) can be expressed in the new variables as

$$\dot{\chi}_i = M_i \chi_i + \left( \sqrt{U_i^2 + V_i^2} h - N_i \eta + \frac{1}{2} \delta \right) \varsigma_i,$$

$$\dot{\varsigma}_i = \left( \sqrt{U_i^2 + V_i^2} h - N_i \eta + \frac{1}{2} \delta \right) \chi_i + M_i \eta \varsigma_i.$$

The characteristic equation of the linear ordinary-differential equation (38) is

$$\lambda^2 - 2M_i \eta \lambda + M_i^2 \eta^2 + \left( N_i \eta - \frac{1}{2} \delta \right)^2 - (U_i^2 + V_i^2) h^2 = 0$$

with the roots

$$\lambda_{1,2} = M_i \eta \pm \sqrt{U_i^2 + V_i^2} h^2 - \left( N_i \eta - \frac{2\omega_j - \omega}{4\omega_j} \right)^2.$$

If the roots have negative real parts, the solutions are stable, and if the real part of at least one of the roots is positive, then the solution is unstable. Therefore stability boundaries in the neighborhoods of $2\omega_j$ can be located in the $(\omega, h)$ plane. In this case, the parametric excitation frequency is near twice the first or second natural frequency. Hence the first or second subharmonic resonance results in instability.

In the case of summation resonance, $k_1 + k_2 = 1$. One can cast the averaged equation in the form

$$\dot{\chi}_1 = (M_{11} \eta + iN_{11} \eta - i\delta k_1) \chi_1 + (U_{11} + iV_{11}) \chi_2,$$

$$\dot{\chi}_2 = (U_{12} + iV_{12}) \chi_1 + (M_{12} + iN_{12} \eta - i\delta k_2) \chi_2,$$

where

$$\chi_1 = a_1 \cos \theta_1 + ia_1 \sin \theta_1, \quad \chi_2 = a_2 \cos \theta_2 - ia_2 \sin \theta_2,$$
The procedure in the previous section can be used to present the stability boundaries in the \( \eta \) numerical obtained. Fig. 1 shows the case that \( \eta, \gamma \) simulations. The Runge–Kutta algorithm is employed to solve numerically the original equation (27). For fixed numerical solution to Eq. (27) can be computed for every point in the \((\omega, h)\) plane. Therefore, a stability boundary can be numerical obtained. Fig. 1 shows the case that \( \eta = 0.0002, \gamma = 8.0 \) and \( \mu = 10.0 \). The comparison indicates that, for enough small \( h \), the analytical results agree excellently with the numerical ones in the first and second subharmonic resonance and the summation resonance, while there is no instability region in the difference resonance. Hence the analytical results are confirmed by numerical calculations.

The effect of the viscosity coefficient on the stability boundaries is examined in Fig. 2, in which \( \gamma = 8.0 \) and \( \mu = 10.0 \). In the subharmonic resonance and the summation resonance, the instability threshold of \( h \) for given \( \omega \), and the smaller instability range of \( \omega \) for given \( h \). Besides, the stability boundary in the summation resonance is most sensitive to the change of the dynamic viscosity, while the stability boundary in the first subharmonic resonance is most insensitive.

The effect of the mean axial speed on the stability boundaries is illustrated in Fig. 3, in which \( \eta = 0.0002 \) and \( \mu = 10.0 \). In the first and second subharmonic resonance and the summation resonance, the stability boundary for \( \gamma = 7.8, 8.0, 8.2 \) are respectively depicted in the dashdot line, the dashed line and the solid line. In the first subharmonic resonance, with the decrease of the mean axial speed, the instability regions drift towards the direction of the increasing \( \omega \). In the second subharmonic resonance, the decreasing mean axial speed results in a slight shrinkage of instability region both in the directions of \( \omega \) and \( h \). In the summation resonance, with the decrease mean axial speed, the instability regions drift towards the directions of the increasing \( \omega \) and \( h \). That is, the smaller mean axial speed results in the larger instability threshold of \( h \) for given \( \omega \), and the

\[
U_{11} = \frac{1}{4} \left( \frac{B_{1,1}^1 - B_{1,2}^{1,2}}{\omega_0} + D_{2,1}^1 + D_{2,2}^{1,2} \right), \quad V_{11} = \frac{1}{4} \left( \frac{B_{1,1}^{1,1} + B_{1,2}^{1,2}}{\omega_0} - D_{2,1}^3 + D_{2,2}^{3,2} \right),
\]

\[
U_{12} = \frac{1}{4} \left( \frac{B_{1,1}^{1,1} - B_{1,2}^{1,2}}{\omega_0} + D_{2,3}^2 + D_{2,4}^{1,4} \right), \quad V_{12} = \frac{1}{4} \left( \frac{B_{1,1}^{2,1} + B_{1,2}^{1,4}}{\omega_0} - D_{2,3}^3 + D_{2,4}^{3,4} \right),
\]

\[
M_{11} = \frac{D_{3,1}^{1,1} + D_{3,2}^{2,2}}{2\omega_0}, \quad N_{11} = \frac{D_{3,1}^{1,2} - D_{3,2}^{1,1}}{2\omega_0}, \quad M_{12} = \frac{D_{3,1}^{3,1} + D_{3,4}^{3,4}}{2\omega_0}, \quad N_{12} = \frac{D_{3,1}^{4,3} - D_{3,4}^{4,4}}{2\omega_0}.
\]

The analysis of the characteristic roots leads to stability boundaries in the neighborhoods of \( \omega_1 + \omega_2 \) can be located in the \((\omega, h)\) plane. Similarly, one can obtain the averaged equation in the case of difference resonance.

4. Numerical results

Both the amplitude \( h \) and the frequency \( \omega \) of axial speed fluctuation play an important part in the stability of responses. The procedure in the previous section can be used to present the stability boundaries in the \((\omega, h)\) plane for a set of given parameters. Numerical results indicate there exist instability regions in the subharmonic resonance and the summation resonance, while there is no instability region in the difference resonance.

First of all, the stability boundaries in the \((\omega, h)\) plane obtained through the averaging method are checked by numerical simulations. The Runge–Kutta algorithm is employed to solve numerically the original equation (27). For fixed \( \eta, \gamma \) and \( \mu \), numerical solution to Eq. (27) can be computed for every point in the \((\omega, h)\) plane. Therefore, a stability boundary can be numerical obtained. Fig. 1 shows the case that \( \eta = 0.0002, \gamma = 8.0 \) and \( \mu = 10.0 \). The comparison indicates that, for enough small \( h \), the analytical results agree excellently with the numerical ones in the first and second subharmonic resonance and the summation resonance, while there is no instability region in the difference resonance. Hence the analytical results are confirmed by numerical calculations.

The effect of the viscosity coefficient on the stability boundaries is examined in Fig. 2, in which \( \gamma = 8.0 \) and \( \mu = 10.0 \). In the subharmonic resonance and the summation resonance, the instability boundary for \( \eta = 0.0, 0.0001, 0.0002 \) are respectively depicted in the dashdot line, the dashed line and the solid line. In all cases, the larger dynamic viscosity leads to the larger instability threshold of \( h \) for given \( \omega \), and the smaller instability range of \( \omega \) for given \( h \). Besides, the stability boundary in the summation resonance is most sensitive to the change of the dynamic viscosity, while the stability boundary in the first subharmonic resonance is most insensitive.

The effect of the mean axial speed on the stability boundaries is illustrated in Fig. 3, in which \( \eta = 0.0002 \) and \( \mu = 10.0 \). In the first and second subharmonic resonance and the summation resonance, the stability boundary for \( \gamma = 7.8, 8.0, 8.2 \) are respectively depicted in the dashdot line, the dashed line and the solid line. In the first subharmonic resonance, with the decrease of the mean axial speed, the instability regions drift towards the directions of the increasing \( \omega \). In the second subharmonic resonance, the decreasing mean axial speed results in a slight shrinkage of instability region both in the directions of \( \omega \) and \( h \). In the summation resonance, with the decrease mean axial speed, the instability regions drift towards the directions of the increasing \( \omega \) and \( h \). That is, the smaller mean axial speed results in the larger instability threshold of \( h \) for given \( \omega \), and the
Fig. 2. Effect of the viscosity coefficient on the stability boundaries. (a) $\omega = 2\omega_1$, (b) $\omega = 2\omega_2$, (c) $\omega = \omega_1 + \omega_2$.

Fig. 3. Effect of the mean axial speed on the stability boundaries. (a) $\omega = 2\omega_1$, (b) $\omega = 2\omega_2$, (c) $\omega = \omega_1 + \omega_2$.

Fig. 4. Effect of the tension on the stability boundaries. (a) $\omega = 2\omega_1$, (b) $\omega = 2\omega_2$, (c) $\omega = \omega_1 + \omega_2$.

larger instability threshold and the smaller instability range of $\omega$ for given $h$. Besides, the stability boundaries in the second subharmonic resonance is not so sensitive to the variations of the mean speed as those in the first subharmonic resonance.

The effect of the tension on the stability boundaries is shown in Fig. 4, in which $\eta = 0.0002$ and $\gamma = 8.0$. In the first and second subharmonic resonance and the summation resonance, the stability boundary for $\mu = 9.8, 10.0, 10.2$, are respectively depicted in the dashdot line, the dashed line and the solid line. Comparing Fig. 4 with Fig. 3, one finds that the effect of the decreasing tension is very similar to that of the increasing mean axial speed.
In all figures, the instability range of $\omega$ increases with the growth of $h$. The smallest instability threshold appears in the first subharmonic resonance, and the largest instability threshold occurs in the summation resonance.

5. Conclusions

This paper treats dynamic stability of an axially accelerating viscoelastic beam. The method of averaging is applied to the 2-term Galerkin truncation of the governing equation. Numerical simulations are presented to confirm and demonstrate the analytical results. From the analytical and numerical work, the following conclusions can be drawn.

(1) There exist instability regions in the subharmonic resonance and the summation resonance, while there is no instability region in the difference resonance. In all cases that the instability occurs, the instability range of the axial speed fluctuation frequency increases with the growth the axial speed fluctuation amplitude.

(2) In the first subharmonic resonance, the increase of the mean axial speed and the decrease of the tension lead to the increase of instability threshold of the axial speed fluctuation frequency, and the increase of the viscosity coefficient leads to a slight increase of instability threshold of the axial speed fluctuation amplitude.

(3) In the second subharmonic resonance, the increasing viscosity coefficient results in the increasing instability threshold of the axial speed fluctuation amplitude, and either the increase of the mean axial speed or the decrease of the tension makes the instability regions reduce slightly.

(4) In the summation resonance, the increase of the viscosity coefficient brings about a dramatic increase of the instability threshold of the axial speed fluctuation amplitude, and either the increase of the mean axial speed or the decrease of the tension leads to the increasing instability thresholds of the axial speed fluctuation amplitude and frequency.

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