Peakons and periodic cusp wave solutions in a
generalized Camassa–Holm equation

Lijun Zhang a,b, Li-Qun Chen b,c,*, Xuwen Huo d

School of Science, Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China
Department of Mechanics, Shanghai University, Shanghai 200444, China
School of Information and Electronic, Zhejiang Sci-Tech University, Hangzhou 310018, China

Abstract

By using the bifurcation theory of planar dynamical systems to a generalized Camassa–Holm equation

\[ m_t + c_0u_x + um_x + 2m u_x = -\gamma u_{xxx} \]

with \( m = u - \frac{x^2}{2} u_{xx} \), \( \alpha \neq 0 \), \( c_0, \gamma \) are constant, which is called CH-r equation, the existence of peakons and periodic cusp wave solutions is obtained. The analytic expressions of the peakons and periodic cusp wave solutions are given and numerical simulation results show the consistence with the theoretical analysis at the same time.

1. Introduction

In 1993, Camassa and Holm [1] derived a shallow water equation

\[ u_t + 2ku_x - u_{xxx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1} \]

which is called the Camassa–Holm equation (CH equation). For the special case \( k = 0 \), they showed that it has peaked solitary wave solutions

\[ u(x, t) = c \exp(-|x - ct|), \tag{1.2} \]

which were called peakons due to the discontinuity of the first derivative at the wave peak.

Recently, many authors have successfully studied the CH equation and some generalized CH equation by using various methods [2–7,10,12]. In [3], Cooper and Shepard derived an approximate solitary wave solution to the CH equation using some variational functions. For general \( k \), Boyd [2] utilized a perturbation series which converges even at the peakon limit to obtain three analytical representations for the spatially periodic generalization of the peakons, the so-called “coshoidal waves”. Just as Boyd noted in [3], that the CH equation has been caused great interest for two reasons. First, it is a model for small amplitude, shallow water waves, just as consistent in this limit as the KdV equation,
which has been intensively studied for a century. Second, the equation is of great interest in the study of solitary waves since it is exactly integrable, its solutions, including multiple solitons are very simple, and it is novel in that its solitary waves have a discontinuous first derivative in contrast to the smoothness of most previously known species of solitary waves. By using the bifurcation methods of the phase portraits of the corresponding planar dynamical system, Liu and Qian [4,5] have given the peakons of the form

\[ u(x, t) = (3k/2) \exp(-|x - kt/2|) - k \]  

(1.3)

when \( k \neq 0 \). Recently, Liu [6] has derived the peakons of the form

\[ u(x, t) = (k + c) \exp(-|x - ct|) - k. \]  

(1.4)

Recently, Dullin [14] and Holm [15] have derived a generalized CH equation

\[ m_t + c_0 u_x + um_x + 2mu_x = -\gamma u_{xxx}, \]  

(1.5)

which is called CH-r equation, where \( m = u - x^2 u_{xxx}, \alpha, \gamma \) are constant and \( \alpha \neq 0 \). Obviously, (1.5) can be rewritten as

\[ u_t + c_0 u_x + 3mu_x - x^2(u_{xxx} + uu_{xxx} + 2u_x u_{xxx}) + \gamma u_{xxx} = 0. \]  

(1.6)

When \( x^2 = 1, c_0 = 2k \) and \( \gamma = 0 \), CH-r equation just is CH equation. Guo and Liu [10] employed phase plane analysis to investigate the peakons of the CH-r equation. They showed that for \( \alpha \neq 0, c, \gamma \) (1.6) has one peakon of the form

\[ u(x, t) = \left( c + \frac{\gamma}{x^2} \right) \exp\left( -\frac{|x - ct|}{|x|} \right) \]  

(1.7)

when \( c_0 = -\frac{3}{2} \) and one peakon of the form

\[ u(x, t) = \left( c + \frac{\gamma}{x^2} \right) \left[ 3 \exp\left( -\frac{|x - ct|}{|x|} \right) - 2 \right] \]  

(1.8)

when \( c_0 = 4c + 3\frac{\gamma}{x^2} \). They derived the two kinds of peakons in three different ways.

In this paper, we will prove that for any parameter \( \alpha \neq 0, c_0, \gamma \) and constant wave speed \( c \), Camassa–Holm Eq. (1.5) has peakons of the form

\[ u(x, t) = -\frac{1}{2} \left( c_0 + \frac{\gamma}{x^2} \right) + \frac{1}{2} \left( 2c + c_0 + 3 \frac{\gamma}{x^2} \right) e^{-\frac{\gamma x}{2}}. \]  

(1.9)

Obviously, we can find that (1.3), (1.4) and (1.7), (1.8) are only special cases of our results.

This paper is organized as follows. In Section 2, we obtain the bifurcation sets and phase portraits of the traveling wave solutions of the CH-r equation. In Section 3, we consider the existence of the smooth solitary and periodic wave solutions of (1.5). We prove the existence of the peakons and the periodic cusp wave solutions of (1.5) and derive its explicit analytic expressions in Section 4. We give the numerical simulation results that show the consistence with the theoretical analysis at the same time.

2. Bifurcation sets and phase portraits of the traveling wave solutions of (1.5)

It is well known that a traveling wave solution of the Eq. (1.6) with constant wave speed \( c \) is a solution of the form

\[ u(x, t) = \phi(x - ct) = \phi(\xi) \]  

with \( \xi = x - ct \). Substituting \( u(x, t) = \phi(x - ct) = \phi(\xi) \) into (1.6), it gives rise to the following ordinary differential equation

\[ \left( c - c_0 \right) \phi' + 3\phi \phi'' - x^2 (\phi \phi''' + 2 \phi' \phi'' - c \phi'''') + \gamma \phi''' = 0, \]  

(2.1)

where \( \cdots \cdots \) denotes the derivative with respect to \( \xi \). Integrating (2.1) once with respect to \( \xi \) leads to

\[ (\gamma + x^2 c - x^2 \phi) \phi'' = (c - c_0) \phi + \frac{3}{2} \phi'^2 + \frac{\phi^2 (\phi')^2}{2} + g, \]  

(2.2)

where \( g \) is an integral constant.

Let \( y = \phi' \), then Eq. (2.2) is equivalent to the two-dimensional system as follows:

\[ \frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(c - c_0) \phi - \frac{3}{2} \phi'^2 + \frac{\phi^2}{2}}{\gamma + x^2 c - x^2 \phi} \]  

(2.3)

except for the solutions of (2.2) which satisfy \( \gamma + x^2 c - x^2 \phi = 0 \) at some point \( \xi \).
Let \( \psi = \phi - \left( \frac{y}{x^2} + c \right) \) (2.4)
then system (2.3) is reduced to

\[
\frac{d\psi}{dz} = y, \quad \frac{dy}{dz} = \frac{b\psi - \frac{1}{2}y^2 + \frac{x^2}{2} + \bar{g}}{-x^2\psi},
\]

where \( b = -2c - c_0 - 3\frac{y}{x^2} \),

\[
\bar{g} = g + (c - c_0)\left( \frac{y}{x^2} + c \right) - \frac{3}{2} \left( \frac{y}{x^2} + c \right)^2.
\]

To simplify our study, it is more convenient to use the transformation \( d\zeta = -x^2\psi d\zeta \), and then Eq. (2.5) becomes

\[
\frac{d\psi}{dz} = -x^2\psi y, \quad \frac{dy}{dz} = b\psi - \frac{3}{2}y^2 + \frac{x^2}{2} + \bar{g},
\]

which is a Hamiltonian system with Hamiltonian

\[
H(\psi, y) = -x^2\psi y^2 - b\psi^2 + \psi^3 - 2\bar{g}\psi = h.
\]

Clearly, the first integral of the system (2.5) is the same as the Hamiltonian of the system (2.7). Consequently, system (2.5) has the same topological phase portraits as system (2.7) except for the straight line \( \psi = 0 \). For the new system (2.7), \( \psi = 0 \) is an invariant straight line solution.

Note that for a fixed \( h \), (2.8) determines a set of invariant curves of (2.7) which contains two or three different branches of curves. As \( h \) is varied, (2.8) determined different families of orbits of (2.7) having different dynamical behaviors.

Now we consider the singular points and their properties of system (2.7). Let

\[
f(\psi) = -\frac{3}{2}\psi^2 + b\psi + \bar{g} = 0, \quad g(y) = \frac{x^2}{2}y^2 + \bar{g}.
\]

To investigate the equilibrium points of (2.7), we need to find all zeros of the function \( f(\psi) \) and \( g(y) \). Let \( M(\psi, y) \) be the coefficient matrix of the linearized system of (2.7) at the equilibrium point \((\psi, y)\). At this equilibrium point, we have

\[
J(\psi, y) = \det M(\psi, y) = -x^4\psi^2 + x^2\psi_y(y - 3\psi).
\]

By the theory of planar dynamical system (see [8,9]), for an equilibrium point of a planar Hamiltonian system, if \( J > 0 \), then it is a center point; if \( J < 0 \), then this equilibrium point is a saddle point; if \( J = 0 \) and the Poincaré index of the equilibrium point is \( 0 \), then this equilibrium point is a cusp. We use the following lemma to state the properties of the equilibrium points of system (2.7).

**Lemma 1.** Let

\[
\psi_{\pm} = \frac{b \pm \sqrt{\Delta}}{3}, \quad y_{\pm} = \frac{\sqrt{-2\bar{g}}}{|x|},
\]

where \( \Delta = b^2 + 6\bar{g} \).

**Case I.** Suppose \( b = -2c - c_0 - 3\frac{y}{x^2} \neq 0 \), i.e. the wave speed \( c \neq \frac{y}{x^2} - \frac{3y}{2x^2} \).

1. When \( -\frac{y}{c} < \bar{g} < 0 \), system (2.7) has four critical points \( A_{\pm}(\psi_{\pm}, 0), B_{\pm}(0, y_{\pm}) \). \( B_{\pm} \) are two saddle points. \( A_{\pm} \) is a saddle point and \( A_{\pm} \) is a center point when \( b > 0 \); \( A_{\pm} \) is a center point and \( A_{\pm} \) is a saddle point when \( b < 0 \).
2. When \( \bar{g} = 0 \), system (2.7) has only two critical points \( O(0, 0) \) and \( A_1(\frac{y}{c}, 0) \). And \( O \) is a cusp and \( A_1 \) is a saddle point whenever \( b > 0 \) or not.
3. When \( \bar{g} > 0 \), system (2.7) has two saddle points \( A_{\pm}(\psi_{\pm}, 0) \).
4. When \( \bar{g} = -\frac{y}{c} \), system (2.7) has two saddle points \( B_{\pm}(0, y_{\pm}) \) and a cusp \( A_1(\frac{y}{c}, 0) \).
5. When \( \bar{g} < -\frac{y}{c} \), system (2.7) has only two saddle points \( B_{\pm}(0, y_{\pm}) \) on y-axis.

**Case II.** Suppose \( b = -2c - c_0 - 3\frac{y}{x^2} = 0 \), i.e. the wave speed \( c = \frac{y}{x^2} - \frac{3y}{2x^2} \).

1. When \( \bar{g} > 0 \), system (2.7) has two saddle points \( A_{\pm}(\psi_{\pm}, 0) \).

1. When \( \bar{g} > 0 \), system (2.7) has two saddle points \( A_{\pm}(\psi_{\pm}, 0) \).
When \( g = 0 \), system (2.7) has only a cusp \( O(0,0) \).

When \( g < 0 \), system (2.7) has two saddle points \( B_{\pm}(0,y_{\pm}) \).

From Lemma 1, we can see that the system (2.7) has bounded solutions only when \(-\frac{b_2^2}{6} < g < 0\). We point out that here we are considering a physical model where only bounded traveling waves are meaningful. Consequently, we only pay attention to the bounded solutions when \(-\frac{b_2^2}{6} < g < 0\).

At the critical points \( A_{\pm} \) and \( B_{\pm} \), the Hamiltonian \( H(\psi, y) \), respectively, are

\[
H(\psi, y) = \psi[-x^2 + (\psi - \psi_{\pm})^2].
\]

Letting \( h_1^+ = h_0 = 0 \) when \( b > 0 \) and \( h_1^- = h_0 = 0 \) when \( b < 0 \), we obtain the following bifurcation curve of the phase portraits on the parameter \( (b, \tilde{g}) \)-plane, namely \( \tilde{g} = -\frac{b_2^2}{6} \). It is easy to see that when \( \tilde{g} = -\frac{b_2^2}{6} \),

\[
H(\psi, y) = \psi[-x^2 + (\psi - \psi_{\pm})^2].
\]

According to the bifurcation theory and above analysis and Lemma 1, we obtain the following Lemma 2 on the bifurcation of phase portraits of system (2.7) as shown in Fig. 1. Note that Eq. (2.5) has the same topological phase portraits except for the straight line \( \psi = 0 \).

**Lemma 2.** For system (2.7), in \((b, \tilde{g})\) parameter plane, there exist three parametric bifurcation curves.

- \( L_1 : \tilde{g} = -\frac{b_2^2}{6} \)
- \( L_2 : \tilde{g} = -\frac{b_2^2}{8} \)
- \( L_3 : \tilde{g} = 0 \)

These curves partition the \((b, \tilde{g})\) parameter plane into four regions denoted by \((A_1)\)–\((A_4)\) shown in Fig. 1. The phase portraits in each region are shown respectively in Figs. 2.1 and 2.2.

3. Smooth solitary and periodic traveling wave solutions of (1.5)

In this section, we consider the existence of smooth traveling wave solutions of (1.5). We first notice that system (2.7) has the same orbits as system (2.5) except for the straight line \( \psi = 0 \). The transformation of variables \( d \xi = \frac{d \psi}{\sqrt{2}} \) only derives the difference of the parametric representations and the direction (when \( \psi > 0 \)) of orbits of the system (2.5) and
Fig. 2.1. Phase portrait of Eq. (2.7) when $b > 0$. (1) $(b, \tilde{g}) \in L_1$. (2) $(b, \tilde{g}) \in L_2$. (3) $(b, \tilde{g}) \in L_3$. (4) $(b, \tilde{g}) \in A_1$. (5) $(b, \tilde{g}) \in A_2$. (6) $(b, \tilde{g}) \in A_3$.

Fig. 2.2. Phase portrait of Eq. (2.7) when $b < 0$. (1) $(b, \tilde{g}) \in L_1$. (2) $(b, \tilde{g}) \in L_2$. (3) $(b, \tilde{g}) \in L_3$. (4) $(b, \tilde{g}) \in A_1$. (5) $(b, \tilde{g}) \in A_2$. (6) $(b, \tilde{g}) \in A_3$. 
(2.7) when \( \psi \neq 0 \). If the orbit of (2.7) has no intersection point with the straight line \( \psi = 0 \), then \( \psi' \) is well defined in (2.5). It follows that on \((\psi, y)\) plane the profile defined by this orbit is smooth. If an orbit of (2.7) intersects with \( \psi = 0 \) at some point \((\psi_0, y_0)\), then the point should be an equilibrium point. Since we only pay attention to the bounded solutions of the system (2.7), we consider only the orbits when \(-\frac{b_0^2}{b} < \tilde{g} < 0\).

We see from (2.6) in Section 2 that
\[
g = g_1 \equiv -\frac{1}{4}(c - c_0)\left(\frac{y}{\bar{x}_1} + c\right) + \frac{3}{8}\left(\frac{y}{\bar{x}_1} + c\right)^2 - \frac{1}{8}(c - c_0)^2
\]
for \((b, \tilde{g}) \in L_1\), namely \( \tilde{g} = -\psi_0^2 \);
\[
g = g_2 \equiv -\frac{1}{6}(c - c_0)^2
\]
for \((b, \tilde{g}) \in L_2\), namely \( \tilde{g} = -\psi_0^2 \);
\[
g = g_3 \equiv -(c - c_0)\left(\frac{y}{\bar{x}_1} + c\right) + \frac{3}{2}\left(\frac{y}{\bar{x}_1} + c\right)^2
\]
for \((b, \tilde{g}) \in L_3\), namely \( \tilde{g} = 0 \). By above analysis and Lemma 2, we can easily get the following conclusions.

**Theorem 3.1.** Suppose that \( b > 0 \), i.e. the wave speed \( c > \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2\tilde{g}} \).

1. When \( g = g_1 \), i.e. \((b, \tilde{g}) \in L_1\) in Fig. 1, for \( h \in (0, h_1^-) \) in (2.8), there are uncountably infinite many smooth periodic traveling wave solutions of (2.7) (see Fig. 4.1(a)).
2. When \( g_1 < g < 0 \), i.e. \((b, \tilde{g}) \in A_1\) in Fig. 1, for \( h \in (0, h_1^-) \) in (2.8), there are uncountably infinite many smooth periodic traveling wave solutions of (2.5) (see Fig. 4.2 (a)).
3. When \( g_2 < g < g_1 \), i.e. \((b, \tilde{g}) \in A_2\) in Fig. 1, for \( h \in (h_1^-, h_1^+) \) in (2.8), there are uncountably infinite many smooth periodic traveling wave solutions of (2.5). As \( h \) decreases from \( h_1^- \) and approaches to \( h_1^+ \), the periods of these periodic traveling waves increase and tend to infinitely great, namely, corresponding to \( h = h_1^+ \) in (2.8), Eq. (2.5) has a smooth solitary traveling wave solutions of valley form (see Fig. 3.1).

By Eq. (2.8), the homoclinic orbit which is corresponding to \( h = h_1^+ \) can be represented as
\[
y^2 = \frac{(\psi - \psi_0^+)^2(\psi - \psi_0^-)}{\bar{x}_1^2},
\]
where
\[
\psi_0^+ = \frac{b - 2\sqrt{h^2 + 6\tilde{g}}}{3}.
\]
Substituting (3.4) into the first equation of system (2.5) and integrating it along the homoclinic orbit, we have
\[
\frac{(z + 1)^2(\psi_+ - \psi)^p(z(\psi_0^+ + \beta))^2(2\psi_0^+)^{(p-1)}(z(\psi_0^+) + 1)^2(\psi_+ - \psi_0^+)^{p-1}}{(z + \beta)^{(2p+1)}} = e^{-|z|},
\]

Fig. 3.1. From smooth periodic traveling wave evolves to smooth solitary wave as \( h \) decreases from \( h_1^- \) and approaches to \( h_1^+ \) in the case \( b > 0 \) and \( g_2 < g < g_1 \), i.e. \((b, \tilde{g}) \in A_2\). (a) \( \psi(0) = 0.5, y(0) = 0 \); (b) \( \psi(0) = 0.28, y(0) = 0 \); (c) \( \psi(0) = 0.2697, y(0) = 0 \).
where
\[ z(\psi) = \sqrt{\frac{\psi}{\psi + \psi_0}} \quad \text{and} \quad \beta = \sqrt{\frac{\psi}{\psi + \psi_0}}. \]  
(3.7)

(3.6) is the analytic expression of the smooth solitary wave solution of (1.5).

**Theorem 3.2.** Suppose that \( b < 0 \), i.e. the wave speed \( c = \frac{\omega}{2} - \frac{\beta}{2\pi} \).

1. When \( g = g_1 \), i.e. \( (b, \tilde{g}) \in L_1 \) in Fig. 1, for \( h \in (h_1^+, 0) \) in (2.8), there are uncountably infinite many smooth periodic traveling wave solutions of (2.5) (see Fig. 4.3(a)).

2. When \( g_1 < g < 0 \), i.e. \( (b, \tilde{g}) \in A_1 \) in Fig. 1, for \( h \in (h_1^+, 0) \) in (2.8), there are uncountably infinite many smooth periodic traveling wave solutions of (2.5) (see Fig. 4.4(a)).

3. When \( g_2 < g < g_1 \), i.e. \( (b, \tilde{g}) \in A_2 \) in Fig. 1, for \( h \in (h_1^+, h_1^-) \) in (2.8), there are uncountably infinite many smooth periodic traveling wave solutions of (2.5). As \( h \) decreases from \( h_1^+ \) and approaches to \( h_1^- \), the periods of these periodic traveling waves increase and tend to infinitely great, namely, corresponding to \( h = h_1^- \) in (2.8), Eq. (2.5) has a smooth solitary traveling wave solutions of peak form (see Fig. 3.2).

By Eq. (2.8), the homoclinic orbit which is corresponding to \( h = h_1^- \) can be represented as
\[ y^2 = \frac{(\psi - \psi_0^2)(\psi - \psi_0^2)}{\sqrt{2\beta}}. \]  
(3.8)

where
\[ \psi_0^2 = \frac{b + 2\sqrt{b^2 + 6g}}{3}. \]  
(3.9)

Substituting (3.4) into the first equation of system (2.5) and integrating it along the homoclinic orbit, we have
\[ \frac{(z + 1)^2(\psi_0^2 - \psi_0^2)(z + 1)^2(\psi_0^2 - \psi_0^2)}{(z + 1)^2(\psi_0^2 - \psi_0^2)(z + 1)^2(\psi_0^2 - \psi_0^2)} = e^{-\psi_0^2}, \]  
(3.10)

where
\[ z(\psi) = \sqrt{\frac{\psi}{\psi + \psi_0}} \quad \text{and} \quad \beta = \sqrt{\frac{\psi}{\psi + \psi_0}}. \]  
(3.11)

(3.6) is the analytic expression of the smooth solitary wave solution of (1.5).

**Remark.** When \( b = 0 \), i.e. the wave speed \( c = \frac{\omega}{2} - \frac{\beta}{2\pi} \) the system (2.7) has only two saddle points on \( \psi \)-axis. Consequently, system (2.5) has no bounded solution when the wave speed \( c = \frac{\omega}{2} - \frac{\beta}{2\pi} \).

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Fig. 3.2. From smooth periodic traveling wave evolves to solitary wave as \( h \) increases from \( h_1^+ \) and approaches to \( h_1^- \) in the case \( b < 0 \) and \( g_2 < g < g_1 \) i.e. \( (b, \tilde{g}) \in A_2 \). (a) \( \psi(0) = 0.5, \gamma(0) = 0 \); (b) \( \psi(0) = -0.28, \gamma(0) = 0 \); (c) \( \psi(0) = -0.2697, \gamma(0) = 0 \).
4. The existence and the analytic expressions of the peakons and the periodic cusp wave solutions of (1.5)

In this section, we shall point out that the existence of the straight lines $\psi = 0$ in the $(\psi, y)$-phase plane of the system (2.5) is the original reason for the appearance of non-smooth traveling wave solutions. We also derive the analytic expressions of the peakons and the periodic cusp wave solutions of (1.5).

We see from Figs. 2.1 and 2.2 that when $(b, \tilde{g}) \in L_1, A_1$, in Fig. 1 for $h < h_1^-$ when $b > 0$ (and $h > h_1^+$ when $b < 0$), as $h$ decreases and approaches to 0, a segment of the left (right) arcs of the orbits of periodic family $\{I^n_h\}$ surrounding the center $A_-(\psi, 0)$ ($A_+(\psi, 0)$) will accumulate into a segment on the straight line $\psi = 0$. Consequently, we have the following conclusion (as proved in [11,13]).

Suppose that the parameter group $(b, \tilde{g})$ of (2.5) satisfies the condition $(b, \tilde{g}) \in L_1, A_1$ see Fig. 3.2 .

Case I. Suppose that $b > 0$, i.e. $c > -\frac{a}{2} - \frac{3\tilde{g}}{2}$.

(1) When $g = g_1$, i.e. $(b, \tilde{g}) \in L_1$ in Fig. 1, for $h \in (0, h^-_1)$ in (2.8), Eq. (2.5) has uncountably infinite many periodic traveling wave solutions; When $h$ decreases from $h^-_1$ and approaches to 0, these periodic traveling waves gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves and the periods of the waves increase and tend to infinitely great, namely, the limit of the periodic waves is a peakon of valley form (see Fig. 4.1).

(2) When $g_1 < g < 0$ i.e. $(b, \tilde{g}) \in A_1$ in Fig. 1, for $h \in (0, h^-_1)$ in (2.8), Eq. (2.5) has uncountably infinite many periodic traveling wave solutions; When $h$ decreases from $h^-_1$ and approaches to 0, these periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves and the periods of wave increase and tend to a constant (see Fig. 4.2).

Let $I^n_h$ be the limiting curve of the periodic orbit when $h$ approaches to 0. By Eq. (2.8) we can see that $I^n_h$ consists of

$$y^2 = \frac{\psi^2 - b\psi - 2\tilde{g}}{x^2} \quad \text{for} \ 0 < \psi \leq \psi^-_1,$$

(4.1)

Fig. 4.1. From smooth periodic traveling wave evolves to peakon as $h$ decreases from $h^-_1$ and approaches to 0 in the case $b > 0$ and $g = g_1$, i.e. $(b, \tilde{g}) \in L_1$. (a) $\psi(0) = 0.3, y(0) = 0$; (b) $\psi(0) = 10^{-3}, y(0) = 0$; (c) $\psi(0) = 10^{-7}, y(0) = 0$.

Fig. 4.2. From smooth periodic traveling wave evolves to peakon cusp wave as $h$ decreases from $h^-_1$ and approaches to 0 in the case $b > 0$ and $g_1 < g < 0$ i.e. $(b, \tilde{g}) \in A_1$. (a) $\psi(0) = 0.3, y(0) = 0$; (b) $\psi(0) = 10^{-3}, y(0) = 0$; (c) $\psi(0) = 10^{-7}, y(0) = 0$. 
where
\[
\psi' = \frac{b - 2\sqrt{b^2 + 8g}}{2}
\]  
and \( \psi = 0 \), for \( |\psi| \leq \psi_\ast \). Substituting (4.1) into the first equation in (2.5), we have
\[
\int_{0}^{\psi} \frac{d\psi}{\sqrt{\psi^2 - b\psi - 2g}} = \int_{0}^{\psi} \frac{dz}{|z|} \quad \text{for} \quad \zeta > 0
\]  
and
\[
\int_{0}^{\psi} \frac{d\psi}{\sqrt{\psi^2 - b\psi - 2g}} = \int_{0}^{\psi} \frac{-d\zeta}{|\zeta|} \quad \text{for} \quad \zeta < 0.
\]  
From (4.3) and (4.4), we have
\[
\psi(\zeta) = \frac{b}{2} - \frac{1}{4} \left[ (b + 2\sqrt{-2g})\text{e}^{-|\psi|} - (b - 2\sqrt{-2g})\text{e}^{|\psi|} \right] \quad \text{for} \quad 0 < \psi \leq \psi_\ast.
\]  
From (2.4), (4.5) is equivalent to
\[
\phi(\zeta) = -\frac{c_0 + \frac{\gamma}{a^2}}{2} - \frac{1}{4} \left[ (b + 2\sqrt{-2g})\text{e}^{-|\psi|} + (b - 2\sqrt{-2g})\text{e}^{|\psi|} \right].
\]  
Let
\[
T = \zeta(\psi_\ast) = \frac{|\zeta|}{2} \ln \left| \frac{b + 2\sqrt{-2g}}{b - 2\sqrt{-2g}} \right|
\]  
Then
\[
u(x, t) = \psi(x - ct - 2nT) \quad \text{for} \quad (2n - 1)T < x - ct < (2n + 1)T
\]  
is a periodic cusp wave solution to (1.5) with 2T period. Clearly, \( T \) approaches to infinitely great as \( g \) tends to \( -\frac{b^2}{8} \) and \( u(x, t) \) in (4.8) approaches to
\[
u(x, t) = -\frac{1}{2} \left( c_0 + \frac{\gamma}{a^2} \right) + \frac{1}{2} \left( 2c + c_0 + 3 \frac{\gamma}{a^2} \right) \text{e}^{-|\psi_\ast|}
\]  
which is a peakon of the CH-r equation.

Case II. Suppose that \( b < 0 \) i.e. the wave speed \( c < -\frac{\sqrt{3}}{2} \).

(1) When \( g = g_1 \), i.e. \((b, \tilde{g}) \in L_1\), Fig. 1, for \( h \in (h_1^+, 0) \) in (2.8), Eq. (2.5) has uncountably infinite many periodic traveling wave solutions; When \( h \) increases from \( h_1^+ \) and approaches to 0, these periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves and the periods of wave increase and tend to infinitely great, namely the limit curve of the periodic waves is a peakon of peak form (see Fig. 4.3).
is a periodic cusp wave solution to (1.5) with 2
which is a peakon of the CH-r Eq. (1.5). We also can get the solution (4.17) in the following two different ways.

Let \( I^h_+ \) be the limiting curve of the periodic orbit when \( h \) approaches to 0. By Eq. (2.8) we can see that \( I^h_+ \) consists of
\[
y^2 = \frac{\psi^2 - b\psi - 2\tilde{g}}{x^2} \quad \text{for} \quad \psi_1^+ < \psi \leq 0.
\]
where \( \psi_1^+ = \frac{b - \sqrt{b^2 + 4\tilde{g}}}{2} \) and \( \psi = 0 \), for \( |y| \leq y_+ \). Substituting (4.1) into the first equation in (2.5), we have
\[
\int_0^\psi \frac{d\psi}{\sqrt{\psi^2 - b\psi - 2\tilde{g}}} = \int_0^\xi \frac{d\xi}{|x|} \quad \text{for} \quad \xi > 0
\]
and
\[
\int_0^\psi \frac{d\psi}{\sqrt{\psi^2 - b\psi - 2\tilde{g}}} = -\int_0^\xi \frac{d\xi}{|x|} \quad \text{for} \quad \xi < 0.
\]
From (4.3) and (4.4), we have
\[
\psi(\xi) = \frac{b}{2} + \frac{1}{4} \left[ (-b + 2\sqrt{-2\tilde{g}}) e^{-\psi} - (b + 2\sqrt{-2\tilde{g}}) e^{\psi} \right] \quad \text{for} \quad \psi_1^+ < \psi \leq 0.
\]
From (2.4), (4.5) is equivalent to
\[
\phi(\xi) = -\frac{c_0 + y}{2} + \frac{1}{4} \left[ (-b + 2\sqrt{-2\tilde{g}}) e^{-\psi} - (b + 2\sqrt{-2\tilde{g}}) e^{\psi} \right].
\]
Let
\[
T = \frac{\xi}{\psi_1^+} = \frac{|x|}{2} \ln \left| \frac{b - 2\sqrt{-2\tilde{g}}}{b + 2\sqrt{-2\tilde{g}}} \right|
\]
Then
\[
u(x, t) = \phi(x - ct - 2nT) \quad \text{for} \quad (2n - 1)T < x - ct < (2n + 1)T
\]
is a periodic cusp wave solution to (1.5) with \( 2T \) period.
Clearly, \( T \) approaches to infinitely great as \( \tilde{g} \) tends to \( -\frac{b^2}{8} \) and \( u(x, t) \) in (4.15) approaches to
\[
u(x, t) = -\frac{1}{2} \left( c_0 + \frac{y}{x^2} \right) + \frac{1}{2} \left( 2c_0 + 3\frac{y}{x^2} \right) e^{-|x|},
\]
which is a peakon of the CH-r Eq. (1.5). We also can get the solution (4.17) in the following two different ways.

Fig. 4.4. From smooth periodic traveling wave evolves to periodic cusp wave as \( h \) increases from \( h_1^+ \) and approaches to 0 in the case \( b < 0 \) and \( g_1 < g < 0 \) i.e. \( (b, \tilde{g}) \in A_1 \). (a) \( \psi(0) = -0.3, y(0) = 0 \); (b) \( \psi(0) = -10^{-3}, y(0) = 0 \); (c) \( \psi(0) = -10^{-7}, y(0) = 0 \).
1. Letting \( \tilde{g} \) approach to \(-\frac{b}{c} \) in (3.6) or (3.10), we can easily get

\[
\psi(\xi) = \frac{b}{2} - \frac{b}{2} e^{-|\xi|},
\]

which is equivalent to (4.17).

2. When \( g = g_1 \), i.e. \((b, \tilde{g}) \in L_1 \) in Fig. (2.1), \( H(\psi, y) = h_1^+ = 0 \) corresponds to two straight lines \( y = \pm \frac{\psi+\phi_+}{c} \) on it is phase portrait which comprise a triangle together with the straight a line \( \Psi = 0 \) when \( b > 0 \). \( H(\psi, y) = h_1^- = 0 \) corresponds to two straight line \( y = \pm \frac{\psi-\phi_-}{c} \) on it is phase portrait which comprise a triangle together with the straight line \( \Psi = 0 \) when \( b < 0 \). Substituting the equations of the straight lines and integrating along the triangle respectively, we immediately obtain the peakon (4.17).

To sum up, from the above discussion, we have

**Theorem 4.1.** For any parameter \( \alpha \neq 0, \gamma, c_0 \) when the wave speed \( c \neq -\frac{\alpha}{2} - \frac{3\gamma}{2c} \), CH-r Eq. (1.5) has peakons of the form

\[
u(x,t) = -\frac{1}{2} \left( c_0 + \frac{\gamma}{c^2} \right) + \frac{1}{2} \left( 2c + c_0 + 3 \frac{\gamma}{c^2} \right) e^{-|x-ct|}.
\]

**Remark.** Obviously, when \( c_0 = -\frac{\gamma}{c} \), (4.19) just is

\[
u(x,t) = \left( c + \frac{\gamma}{c^2} \right) \exp \left( \frac{-|x-ct|}{|c|} \right).
\]

When \( c_0 = 4c + 3 \frac{\gamma}{c} \), (4.19) just is

\[
u(x,t) = \left( c + \frac{\gamma}{c^2} \right) \left( 3 \exp \left( \frac{-|x-ct|}{|c|} \right) - 2 \right).
\]

Consequently, the results in [4,5,10] are only special cases of our results.

**Theorem 4.2.** Denote that \( b = -2c - c_0 - 3 \frac{\gamma}{2c} \). Suppose that \( g_1 < g < 0 \). For any parameter \( \alpha \neq 0, \gamma, c_0 \) and the wave speed \( c \neq -\frac{\alpha}{2} - \frac{3\gamma}{2c} \),

1. when \( c > -\frac{\alpha}{2} - \frac{3\gamma}{2c} \), CH-r Eq. (1.5) has periodic cusp wave solution with 2T period of the form

\[
u(x,t) = \phi(x - ct - 2nT) \quad \text{for} \quad (2n-1)T < x - ct < (2n+1)T
\]

with

\[
\phi(\xi) = -\frac{c_0 + \frac{\gamma}{c}}{2} - \frac{1}{4} \left( (b + 2\sqrt{-2g}) e^{i\xi} + (b - 2\sqrt{-2g}) e^{-i\xi} \right)
\]

and

\[
T = \frac{1}{2} \ln \left[ \frac{|b + 2\sqrt{-2g}|}{|b - 2\sqrt{-2g}|} \right].
\]

2. when \( c < -\frac{\alpha}{2} - \frac{3\gamma}{2c} \), CH-r Eq. (1.5) has periodic cusp wave solution with 2T period of the form

\[
u(x,t) = \phi(x - ct - 2nT) \quad \text{for} \quad (2n-1)T < x - ct < (2n+1)T
\]

with

\[
\phi(\xi) = -\frac{c_0 + \frac{\gamma}{c}}{2} + \frac{1}{4} \left( (-b + 2\sqrt{-2g}) e^{i\xi} - (b + 2\sqrt{-2g}) e^{-i\xi} \right)
\]

and

\[
T = \frac{1}{2} \ln \left[ \frac{|b - 2\sqrt{-2g}|}{|b + 2\sqrt{-2g}|} \right].
\]
5. Conclusion

In this work we have investigated the peakons and the smooth and non-smooth periodic traveling wave solutions of the generalized Camassa–Holm equation by using the bifurcation theory of the planar dynamical systems. First we gave the bifurcation phase portraits of the corresponding traveling wave system and the parameter conditions under which the peakons and the cusp wave periodic solutions exist. Second we derived the more general explicit analytic expressions of the generalized CH equation that contains some known results. At the same time the numerical simulation results show the consistence with the theoretical analysis.

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References