On the transmissibilities of nonlinear vibration isolation system

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Abstract

Transmissibility is a key parameter to quantify the effectiveness of a vibration isolation system. Under harmonic excitation, the force transmissibility of a linear vibration isolation system is defined as the ratio between the amplitude of the force transmitted to the host structure and the excitation force amplitude, and the displacement transmissibility is the ratio between the displacement amplitude of the payload and that of the base. For a nonlinear vibration isolation system, the force or the displacement responses usually have more frequency components than the excitation. For a harmonic excitation, the response may be periodic, quasi-periodic or chaotic. Therefore, the amplitude ratio cannot well define the transmissibility. The root-mean-square ratio of the response to the excitation is suggested to define the transmissibility. The significance of the modified transmissibility is highlighted in a nonlinear two-stage vibration isolation system consisting of two linear spring connected linear vibration isolators with two additional horizontal linear springs. Harmonic balance method (HBM) is applied to determine the responses with the fundamental and third harmonic. Numerical simulations reveal that chaos may occur in the responses. In both cases, the modified transmissibility works while the original definition cannot be applied to chaotic response.

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1. Introduction

Nonlinear vibration isolation has been concerned by many engineers and scientists [1], for that it can achieve a low dynamic stiffness and hence low natural frequency, but at the same time having a low static deflection [2–8]. Thus, nonlinear vibration isolation comes at a cost of complexity, nonlinearity, which is perceived to be worthwhile in some applications, for example, vibration control of the integrated satellite system [9–11].

Transmissibility is always used to quantify the effectiveness of the isolation system and always defined as the form that the ratio of the amplitude of the transmitted force to the excitation force at each concerning frequency [1,2]. And such definition for the linear vibration isolator has been worked early [6,7], even widely application in the nonlinear vibration isolator with its appearance [1–5,9]. However, nonlinear vibration isolation has other harmonics, quasi-periodic response and chaotic motion rather than only fundamental harmonic when it responses to harmonic excitation [4–6]. So traditional
transmissibility with linear form definition is used to nonlinear vibration isolation remains a fundamental limitation, however, with both the single- and two-stage nonlinear vibration isolators.

Some papers have explore the advantages that can be gained by incorporating geometric stiffness nonlinearity into a two-stage isolator to overcome the problems of high static deflection and low roll-off rate at high frequency [12,13]. Lu and Yang [13] incorporates geometric stiffness nonlinearity into a two-stage isolator to overcome the problems of high static deflection and low roll-off rates at high frequency. It has been found that nonlinearity in the lower stage has a profound effect, and significantly improves the effectiveness of the isolation system. In later work, the upper horizontal springs are connected to the secondary mass, rather than to the ground as described in reference [14]. The revised isolation model is employed to investigate the dynamics of the two-stage nonlinear isolation system and to see whether it can outperform other existing systems, it is found that both the force and displacement transmissibility are reduced in the isolation range as the horizontal stiffnesses at both stages are increased [14]. Although the problem of the definition is treated in the nonlinear vibration isolation system, overall the idea could be extended to other vibration controls.

Fig. 1. Schematic of the two-stage nonlinear isolation system. (a) actual system, (b) equivalent lumped parameter model. The mass, \(m_1\) is the suspended (primary) mass and \(m_2\) is the intermediate (secondary) mass.
In this paper, modified transmissibility defined as the root-mean-square (RMS) ratio of the transmitted force to the excitation force is to evaluate the performance of such the two-stage nonlinear vibration isolation system, the results are compared with the linear form definition that only the fundamental harmonic response is considered as in Ref. [14].

2. General transmissibility definition

The target parameter which is used to quantify the effectiveness of the isolation system is the transmissibility. For the conventional single-degree-of-freedom system this parameter is defined as the modulus of the ratio between the force transmitted to the host structure \( \bar{f}_t \) and the excitation force \( f_e \), or, in the case of base excitation, as the ratio between the displacement \( \bar{x}_t \) of the payload and that of the base \( \bar{x}_e \). For the conventional case, the force and displacement transmissibilities are the same and for both the mathematical expression is [6–8]

\[
|T| = \frac{|f|}{|f_e|} = \frac{x_t}{x_e} = \frac{1 + 4\xi^2 \Omega^2}{(1 - \Omega^2)^2 + 4\xi^2 \Omega^2}
\]

where \( \xi \) is the damping ratio and \( \Omega \) is the non-dimensional excitation frequency.

Another definition of transmissibility is the measure used in the investigation, which is defined in terms of the RMS ratio of transmitted force/displacement to the excitation force/displacement. In this case, it only involves the measurement of time response.

\[
|T_f| = \frac{\text{RMS}(f_t(t))}{\text{RMS}(f_e(t))} \quad |T_d| = \frac{\text{RMS}(x_t(t))}{\text{RMS}(x_e(t))}
\]

This is due to the point of the energy conservative, the other frequency components of response are taken into the consideration using RMS, and it could lead to the reduction of the energy loss.

3. A case study: two-stage nonlinear vibration isolation system

3.1. Transmissibility

The nonlinear vibration isolation system of interest in this paper is shown in Fig. 1a. It consists of a primary mass \( m_1 \) suspended on an upper-stage nonlinear isolator which is attached to a secondary mass \( m_2 \) that is connected to the ground by the lower-stage nonlinear isolator. Both isolators have geometrically nonlinear stiffness with hardening characteristics, and linear viscous damping. A simple model of this system is shown in Fig. 1b. Each of the isolators is modeled by two horizontal (lateral) linear springs with stiffness \( k_{h1} \) and \( k_{h2} \), two vertical linear springs with stiffness \( k_{v1} \) and \( k_{v2} \), which support the static load, and two vertical linear dampers with damping coefficient \( c_1 \) and \( c_2 \) [5]. The subscripts 1 and 2 refer to the upper and lower stages respectively. The geometrically nonlinear stiffness is due to the horizontal springs to deform along orthogonal directions. Here, only horizontal springs are considered, which result in an overall hardening stiffness characteristic of the isolator as in [7].

This paper is to investigate the criterion for evaluation of the performance of such isolation system. The model shown in Fig. 1 is similar to that discussed in Ref. [12]. The system can be either excited with a harmonic force \( f_e(t) \) and the base set firmly to ground so that \( x_e = 0 \), or base excited with a displacement \( x_e(t) \) and \( f_e = 0 \).

3.1.1. Force transmissibility

The matrix equation of motion for the system shown in Fig. 1 when it is excited by a harmonic force \( f_e(t) = F_e \cos(\omega t) \) with the system placed on a rigid base so that \( x_e = 0 \), is given by

\[
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}
\]

where,

\[
\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} F_e \cos(\omega t) \\ 0 \end{bmatrix}
\]

\[
\mathbf{K}(\mathbf{x}) = \begin{bmatrix} k_{v1} + 2k_{h1} \left(1 - \frac{l_0}{\sqrt{(x_1 - x_2)^2 + l_0^2}}\right) & -k_{v1} - 2k_{h1} \left(1 - \frac{l_0}{\sqrt{(x_1 - x_2)^2 + l_0^2}}\right) \\ -k_{v1} - 2k_{h1} \left(1 - \frac{l_0}{\sqrt{(x_1 - x_2)^2 + l_0^2}}\right) & k_{v1} + 2k_{h1} \left(1 - \frac{l_0}{\sqrt{(x_1 - x_2)^2 + l_0^2}}\right) + k_{v2} + 2k_{h2} \left(1 - \frac{l_0}{\sqrt{x_2^2 + l_0^2}}\right) \end{bmatrix}
\]

For small displacements of the primary and secondary masses such that \( x_1 < 0.2l_1 \) and \( x_2 < 0.2l_2 \), Eq. (3) can be approximated by two coupled Duffing equations, which can be written in non-dimensional matrix form as

\[
\mathbf{\ddot{x}}^{(3)} + \mathbf{\dot{x}}^{(3)} + \mathbf{K}_d\mathbf{x}^{(3)} = \mathbf{\ddot{f}}
\]
where,
\[
\dot{\mathbf{M}} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad \dot{\mathbf{C}} = 2 \begin{bmatrix} \zeta_1 & -\zeta_1 \\ -\zeta_1 & \zeta_1 + \mu \zeta_2 \end{bmatrix}, \quad \dot{\mathbf{K}}_1 = \begin{bmatrix} \Omega^2_1 & -\Omega_1^2 \\ -\Omega_1^2 & \Omega^2_1 + \Omega^2_2 \end{bmatrix}, \quad \dot{\mathbf{K}}_3 = \begin{bmatrix} \gamma_1 & 0 \\ -\gamma_1 & \gamma_2 \end{bmatrix}.
\]

\[
\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad \dot{\mathbf{X}}^{(3)} = \begin{bmatrix} (\dot{x}_1 - \dot{x}_2)^3 \\ \dot{x}_1^3 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \dot{F}_e \cos(\Omega t) \\ 0 \end{bmatrix}
\]

and \( \dot{x}_1 = \frac{x_1}{x_1}, \dot{x}_2 = \frac{x_2}{x_2}, x_0 = \left( \frac{\omega_1^2 - \Omega^2}{\mu} \right)^{\frac{1}{2}}, \dot{k}_{h1} = \frac{k_{h1}}{x_1}, \dot{k}_{h2} = \frac{k_{h2}}{x_1}, \dot{k}_{v2} = \frac{k_{v2}}{x_2}, a_{0n} = \sqrt{\frac{x_0}{m_0}}, \Omega^2 = 1 - 2 \left( \frac{1}{2} - 1 \right) k_{h1}, \Omega_{v2} = k_{v2} - 2 \left( \frac{1}{2} - 1 \right) k_{h2}, \gamma_1 = \left( \frac{1 - \mu}{\mu} \right) \dot{k}_{h1}, \gamma_2 = \dot{k}_{v2} \frac{k_{h2} - k_{h1}}{k_{h2}}, \mu = \frac{m_0}{m_1}, \zeta_1 = \frac{\zeta_{h1,0}}{m_0 x_0}, \zeta_2 = \frac{\zeta_{h2,0}}{m_0 x_0}, \Omega = \frac{m_0 x_0}{m_1 x_1}, r = a_{0n} t, \dot{F}_e = \frac{F_e}{x_1}, \dot{t} = \frac{t}{x_1}, \) with \((\cdot)' = d(\cdot)/dr\), in which \( x_0 \) is the static displacement of \( m_1 \), when the masses are placed on the springs such that \( k_{h1} \) and \( k_{h2} \) are in the horizontal position as shown in Fig. 1.

The HBM can be used together with the assumption that the vector of non-dimensional displacements has the following form
\[
\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{X}_1 \cos(\Omega t + \varphi_{f1}) \\ \dot{X}_2 \cos(\Omega t + \varphi_{f2}) \end{bmatrix}
\]

The resulting amplitude-frequency matrix equation is similar to that derived in [13,14], and is given by
\[
\begin{bmatrix} \dot{K}_1 - \Omega^2 \dot{M} \end{bmatrix} \mathbf{X}_{f} = 3^3 \dot{K}_3 \begin{bmatrix} \dot{X}^{(3)} \end{bmatrix} + \Omega \partial \mathbf{X} \dot{\mathbf{F}}_f A = \mathbf{F} \mathbf{F}_f
\]

where,
\[
\mathbf{F} = \begin{bmatrix} \cos \varphi_{f1} & \sin \varphi_{f1} \\ \cos \varphi_{f2} & \sin \varphi_{f2} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \dot{\mathbf{X}} = \begin{bmatrix} \dot{X}_1 \\ 0 \\ \dot{X}_2 \end{bmatrix}, \quad \dot{\mathbf{F}}_f = \begin{bmatrix} \dot{F}_e \\ 0 \\ 0 \end{bmatrix}
\]

The force transmitted to the ground is given by
\[
\dot{f}_t = \dot{K}_1 \dot{\mathbf{X}} + \dot{K}_3 \dot{\mathbf{X}}^{(3)} + \dot{\mathbf{C}} \dot{\mathbf{X}}
\]

where,
\[
\dot{K}_1 = \begin{bmatrix} 0 & -2 \dot{k}_{v2} \frac{1}{T} \\ \dot{k}_{v2} - 2 \dot{k}_{h2} \frac{1}{T} \end{bmatrix}, \quad \dot{K}_3 = \begin{bmatrix} 0 & \dot{k}_{v2} \frac{1}{T} \\ \dot{k}_{h2} \frac{1}{T} \end{bmatrix}, \quad \dot{\mathbf{C}} = \begin{bmatrix} 0 & 2 \mu \zeta_2 \end{bmatrix}
\]

Using the HBM, and assuming that the transmitted force is predominantly at the same frequency as the excitation so that \( \dot{f}_t = \dot{F}_t \cos(\Omega t + \varphi_t) \), the magnitude of the force can be determined from
\[
\dot{F}_t \mathbf{F} = \dot{K}_1 \dot{\mathbf{X}} + \dot{K}_3 \dot{\mathbf{X}}^{(3)} + \frac{3}{4} \dot{K}_3 \dot{\mathbf{X}}^{(3)} + \Omega^2 \dot{\mathbf{C}} \dot{\mathbf{X}} \dot{\mathbf{F}}_f A
\]

where, \( \mathbf{F}_f = \begin{bmatrix} \cos \varphi_t & \sin \varphi_t \end{bmatrix} \). The force transmissibility can then be determined by
\[
|T_f| = \frac{\dot{F}_t}{\dot{F}_f}
\]

### 3.1.2. Displacement transmissibility

If there is an imposed harmonic vibration of the base \( x_e(t) = x_e \cos(\omega t) \), rather than force excitation of the mass then displacement transmissibility is of interest. In this case \( f_e = 0 \), and the corresponding non-dimensional matrix equation of motion is given by
\[
\ddot{\mathbf{M}} \ddot{\mathbf{X}} + \dot{\mathbf{C}} \ddot{\mathbf{X}} + \dot{\mathbf{K}}_1 \dot{\mathbf{X}} + \dot{\mathbf{K}}_3 \dot{\mathbf{X}}^{(3)} = 0
\]

where
\[
\ddot{\mathbf{M}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \dot{\mathbf{C}} = 2 \begin{bmatrix} \zeta_1 & -\zeta_1 \\ -\zeta_1 & \zeta_1 + \mu \zeta_2 \\ -\zeta_1 & -\zeta_1 \end{bmatrix}, \quad \dot{\mathbf{K}}_1 = \begin{bmatrix} \Omega^2_1 & -\Omega^2_1 \\ -\Omega^2_1 & \Omega^2_1 + \Omega^2_2 \\ -\Omega^2_1 & -\Omega^2_1 \end{bmatrix}, \quad \dot{\mathbf{K}}_3 = \begin{bmatrix} \gamma_1 & 0 \\ -\gamma_1 & \gamma_2 \end{bmatrix}, \quad \dot{\mathbf{X}}^{(3)} = \begin{bmatrix} \dot{x}_1 - \dot{x}_2 \\ (\dot{x}_1 - \dot{x}_2)^3 \end{bmatrix}, \quad \dot{x}_e = \dot{x}_e \cos(\Omega t + \varphi_{x_e}), \quad \dot{X}_e = \frac{X_e}{X_e}
\]
Assuming that the vector of non-dimensional displacements has the form,

\[
\begin{aligned}
\{\dot{x}_1(t)\} &= \{\dot{X}_1 \cos(\Omega t + \varphi_{X1})\} \\
\{\dot{x}_2(t)\} &= \{\dot{X}_2 \cos(\Omega t + \varphi_{X2})\}
\end{aligned}
\]  

(11)

The resulting amplitude-frequency matrix equation is given by

\[
(\hat{K}_1 - \Omega^2 \hat{M})\hat{X} + \frac{3}{4}\hat{k}_4 (\hat{X}^{(2)}\Phi_\epsilon) + \Omega \hat{C}\hat{X}= 0
\]

(12)

where,

\[
\Phi_\epsilon = \begin{bmatrix}
\cos \varphi_{X1} & \sin \varphi_{X1} \\
\cos \varphi_{X2} & \sin \varphi_{X2} \\
\cos \varphi_{Xe} & \sin \varphi_{Xe}
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}, \quad \hat{X} = \begin{bmatrix}
\dot{X}_1 & 0 & 0 \\
0 & \dot{X}_2 & 0 \\
0 & 0 & \dot{X}_e
\end{bmatrix}.
\]

\[
\hat{X}^{(3)} = \begin{bmatrix}
\dot{X}_1^3 - 2\dot{X}_1\dot{X}_2^2 & 2\dot{X}_1^2\dot{X}_2 - \dot{X}_2^3 & 0 \\
0 & \dot{X}_2^3 - 2\dot{X}_2\dot{X}_e^2 & 2\dot{X}_2^2\dot{X}_e - \dot{X}_e^3
\end{bmatrix}
\]

Then displacement transmissibility can be determined by

\[
|T_D| = \frac{\dot{X}_1}{X_e}
\]

(13)

3.2. Modified transmissibility

Of interest in this paper are the transmissibility of such an isolation system, and the effects of additional higher harmonics on the transmissibility. The model used in this section is the same with model discussed in Section 3.1, but here both fundamental and higher harmonic responses are considered rather than to the only fundamental-harmonic response as was the case in reference [12]. By applying the method of harmonic balance with higher harmonics, the procedure will give more accurate results. In the limit of including all the harmonics, they must give exactly the same solution with exact result. But, it is difficult to construct higher-order analytical approximations to the solution by the method of harmonic balance, especially for two-degree-of-freedom nonlinear system. Therefore, in this paper we restrict our investigation by the third harmonic only. As the authors’ intention is to investigate the difference transmissibility of the isolation system when it has an additional third harmonic responds at the excitation frequency. So the transmissibility of the system in the investigation is defined as Eq. (2) rather than (Eqs. (9) and 13).

3.2.1. Force transmissibility

Accordingly, the assumption of solution that the vector of non-dimensional displacements has been changed with the form

\[
\begin{aligned}
\{\dot{x}_1(t)\} &= \{\dot{X}_{11} \cos(\Omega t + \varphi_{X11}) + \dot{X}_{13} \cos(3\Omega t + \varphi_{X13})\} \\
\{\dot{x}_2(t)\} &= \{\dot{X}_{21} \cos(\Omega t + \varphi_{X21}) + \dot{X}_{23} \cos(3\Omega t + \varphi_{X23})\}
\end{aligned}
\]

(14)

Substituting Eq. (14) into Eq. (4), the resulting amplitude-frequency equation can be written as nonlinear algebraic equations

\[
g_{\ell}(\Omega, \dot{X}_{11}, \dot{X}_{13}, \dot{X}_{21}, \dot{X}_{23}, \varphi_{X11}, \varphi_{X13}, \varphi_{X21}, \varphi_{X23}, \mu, \zeta_1, \zeta_2, \Omega_1, \Omega_2, \gamma_1, \gamma_2, \hat{F}_\ell) = 0, \quad i = 1 \ldots 8
\]

(15)

Because the third harmonic exists, the force transmitted to the ground is given by

\[
\hat{f}_i = \hat{K}_1 \hat{X} + \hat{K}_3 \hat{X}^{(3)} + \hat{C}\hat{X}
\]

(16)

where,

\[
\hat{K}_1 = \begin{bmatrix}
0 & -k_{l2} \frac{1}{T} \\
-k_{l2} \frac{1}{T} & 0
\end{bmatrix}, \quad \hat{K}_3 = \begin{bmatrix}
0 & 0 \\
0 & 2\mu\zeta_2
\end{bmatrix}, \quad \hat{C} = \begin{bmatrix}
0 & \mu\zeta_2
\end{bmatrix}, \quad \hat{X} = \begin{bmatrix}
\dot{X}_{11} \cos(\Omega t + \varphi_{X11}) + \dot{X}_{13} \cos(3\Omega t + \varphi_{X13}) \\
\dot{X}_{21} \cos(\Omega t + \varphi_{X21}) + \dot{X}_{23} \cos(3\Omega t + \varphi_{X23})
\end{bmatrix}
\]

The transmitted force can be written in the form as

\[
\hat{f}_i = \hat{F}_{11} \cos(\Omega t + \varphi_{11}) + \hat{F}_{13} \cos(3\Omega t + \varphi_{13})
\]

(17)
The transmitted force has higher-order harmonics rather than only the fundamental harmonic as the same with the excitation frequency, thus the force transmissibility can then be determined by

\[ |T_F| = \frac{\text{RMS}(\hat{f}_1(t))}{\text{RMS}(\hat{f}_e(t))} \left(\frac{\hat{F}_2^2 + \hat{F}_3^2}{F_e}\right) \] (18)

### 3.2.2. Displacement transmissibility

Substituting Eq. (14) into Eq. (10), the resulting amplitude-frequency matrix equation can be obtained

\[ g_i(\Omega, \hat{X}_{11}, \hat{X}_{12}, \hat{X}_{21}, \hat{X}_{22}, \varphi_{X_{11}}, \varphi_{X_{12}}, \varphi_{X_{21}}, \varphi_{X_{22}}, \mu, \xi_1, \xi_2, \Omega_1, \Omega_2, \gamma_1, \gamma_2, \hat{x}_e) = 0, \quad i = 1 \ldots 8 \] (19)

Then displacement transmissibility can be determined by

\[ |T_D| = \frac{\text{RMS}({\hat{x}_1(t)})}{\text{RMS}(x_e(t))} \left(\frac{\hat{X}_{11}^2 + \hat{X}_{13}^2}{X_e}\right) \] (20)

### 3.3. Discussion

Fig. 2 illustrates of the 1st (black circle points) and 3rd (blue solid line) order approximation HBM solution for force (a, c) and displacement (b, d) transmissibility changing with different horizontal stiffness \( \hat{k}_{h2} \). The parameters are \( i = 0.7, \mu = 0.2, \hat{k}_{h2} = 1, \hat{k}_{h1} = 1, \xi_1 = \xi_2 = 0.01 \). (a) and (b): high level of the excitation, \( \hat{F}_e = F_{e \text{ max}}, \hat{X}_e = X_{e \text{ max}} \) (Reference [13,14]); (c) and (d): low level of the excitation, \( \hat{F}_e = 0.1 F_{e \text{ max}}, \hat{X}_e = 0.7 X_{e \text{ max}} \). The red dotted line is the unstable solutions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
constitutive parameters and the excitation amplitude for the parameters chosen. Of particularly interesting, at certain nonlinearity, amplitude-frequency response curve for the force transmissibility are crossed by themselves when the excitation has low frequency and high amplitude. Fig. 2(a) and (b) illustrate the effects on the force and displacement transmissibility in which the excitation amplitude is fixed to $F_e = F_{e\text{ max}}$ for force transmission, $X_e = X_{e\text{ max}}$ for displacement transmission (Ref. [13,14]), when the stiffness of the horizontal springs is changed. It can be seen that as nonlinearity is increased by increasing $k_{h2}$ both the second natural frequency and the magnitude of the transmissibility at this frequency are reduced. There are good agreements between 1st and 3rd approximation solutions. Fig. 2(c) and (d) shows the effect of that on the transmissibilities when the excitation amplitude is fixed to $F_e = 0.1F_{e\text{ max}}, X_e = 0.7X_{e\text{ max}}$. The same conclusions are achieved. Comparing Fig. 2(c) and (d) with 2(a) and 2(b), respectively, it can be seen that the roll-off rate of the transmissibility at high frequency are the same when the constitutive parameters are the same, regardless of the force or displacement transmissivity, and high or low excitation amplitude.

Fig. 3 shows the 3rd order HBM solution for the force (blue solid line) and displacement (red dashed line) transmissibility when the parameters are the same as in Fig. 2. It is found that force and displacement transmissibility are almost the same, but they are different around the resonance frequency. The force and the acceleration transmissibility would be the same for the two-stage nonlinear isolation system, but the displacement transmissibility in which the base displacement is the same at each frequency is different. The magnitude of displacement transmissibility would only be the same as the one of force transmissibility if the base acceleration is kept the same at each frequency.

An approximate solution for the steady state periodic response is determined by HBM, and then the local stability of this solution is determined by analyzing the linearized equation of motion. An exact stability analysis of the linear variational equation can provide erroneous stability information for the approximate solutions. The consistent stability information about the solutions could be obtained only when the linearized variational equation is analyzed by approximate methods, and the level of accuracy of the stability analysis is consistent with that of the approximate solutions. In this paper, the difference between an approximate and the next higher order stability analysis is used to determine the role of higher harmonics in the periodic response. Then the error procedure is used to ensure the qualitatively correct and numerically accurate of the stability analysis for the approximate solutions [17]. The stability of the force and displacement transmissibility with 3rd approximation solutions are checked, the red dotted line in Fig. 3 is unstable solutions of force and displacement transmissibility. Of interesting, some unstable solutions are found for force transmissibility when the excitation has low frequency and high amplitude, but not for displacement transmissibility at the same frequency region and the same parameters.

The 3rd order approximation HBM solutions of the force and displacement transmissibilities are also plotted in Fig. 3 together with numerical solutions obtained by the fourth-order Runge-Kutta method. It can be seen that there is reasonable agreement and so both the HBM analytical solutions and numerical method by the fourth-order Runge-Kutta method can be used for further investigation of the dynamic behavior.
The additional harmonics has a profound effect on the shape of the transmissibility curve and the time-history response in the concerning frequency region. The former can be seen in Fig. 3 and the latter can be seen in Fig. 4. The effects of the nonlinearity on the time history response curves of the transmitted force at two different excitation frequencies ($\Omega = 0.06$ and $\Omega = 0.08$) are evident. The nonlinearity distorts the time history curves of the transmitted force with the undesirable result that there are many high order harmonic terms in the response.

4. Generalization to aperiodic responses

4.1. Modified transmissibility for quasi-periodic response

For a harmonic excitation, the response may be periodic, quasi-periodic or chaotic. However, as mentioned before, the analytical results and the conclusions based on those results are only strictly applicable for periodic response. When the excitation frequency and the natural frequency of the nonlinear isolation system in Fig. 1(b) are incommensurable, the response may be quasi-periodic, (Eqs. (8) and 12) cannot be used to evaluate such the transmissibility. In this case, the amplitude ratio cannot well define the transmissibility. The root-mean-square ratio of the response to the excitation is suggested to define the transmissibility. Fig. 5 shows the time series of the transmitted and excitation force under the excitation with parameters $\Omega = 0.07$ and $F_e = F_{e_{\text{max}}}$. It can be seen that response of the transmitted force has quasi-periodic characteristic. And the transmissibility can be determined by employing Eq. (2). In this case, when the analytical period $T > \lambda (2\pi / \Omega)^2 (\lambda = 2.3$ is the incommensurability frequency ratio of the fundamental and next higher harmonic for the response) is chosen, the transmissibility is a constant $|T_F| = 1.78\text{dB}$. The reason for
the convergence of the solution for the transmissibility is that the mapping in the Poincare surface of section is a closed curve for
the quasi-periodic response.

4.2. Modified transmissibility for chaotic motion

When the nonlinear isolation system in Fig. 1(b) is subject to relatively higher excitation amplitude, the response may be
chaotic. In this case, the modified transmissibility is used to measure the isolation performance. To determine the trans-
missibilities of the systems for chaotic motion, the fourth-order Runge-Kutta method was used to solve the equations of the
motion directly. At each frequency of excitation the response was calculated numerically and the RMS of the steady-state
response was used to plot the displacement transmissibility, which are shown in Fig. 6 for high excitation levels. It is clear
that the response shows the chaotic motion. The level of the accurate for the transmissibility is increased as the increasing of
the analytical period $T$. Of interesting the transmissibility plot is almost the same at the high analytical period $T$ about more
than 1000$\pi$/$\Omega$ (500 of excitation period). Namely the chaotic transmissibility can only in
finitely approach to a constant and
cannot arrive at it, when the analytical period $T$ is increased. It is also found that the transmissibility using decreasing
frequency is lower than using increasing frequency at the frequency region between the jump-up and jump-down
frequencies.

5. Summary

The traditional linear form transmissibility remains a fundamental limitation for nonlinear vibration isolation system.
The transmissibility has two different definitions: the one is the modulus of the ratio of the transmitted force/displacement
to the excitation force/displacement at the excitation frequency; the other is the RMS ratio of the transmitted force/dis-
placement to the excitation force/displacement. These two transmissibility definitions are applied to the two-stage non-
linear vibration isolation system, and the results are compared. It has been found that the modified transmissibility are
slightly different with the conventional one for both the force and displacement transmission. Of particularly, it has more
different at the frequency region which has the force transmissibility curve crossing by itself. It has been also found that
force and displacement transmissibility with the modified definition are more agreement at high frequencies but they are
different around resonance frequency.
Although the modified definition was proposed for the nonlinear vibration isolation, a similar generalization of the definition could be applied to vibration absorption, energy harvesting, sensing, etc.

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