Nonlinear vibration of axially accelerating hyperelastic beams

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ABSTRACT

This paper investigates principal parametric resonance of axially accelerating hyperelastic beam. Hyperelasticity is integrated into axially moving material for the first time. Based on the continuum mechanics theory, the coupled nonlinear partial differential equations of motion are derived from the extended Hamilton's principle. The model equations are simplified into a single integro-differential equation, which governs the transverse vibration of the hyperelastic beam. The method of multiple scales is used to solve the integro-differential equation to obtain the nonlinear response of the principal parametric resonance. The effect of the material parameter (i.e., in plane Poisson’s ratio) on the type of the nonlinear vibration behavior and amplitude of the nontrivial solutions of steady-state have been investigated. Further, the couple nonlinear governing equations are solved by Galerkin’s method. Comparison between the analytical results and the results of the Galerkin's method are made and good agreement are found.

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1. Introduce

Axially moving beams are widely used in many engineering structures, such as civil, aerospace and automotive applications et al. The nonlinear dynamic behavior of an axially accelerating beam have been received much researches over many years.

It is well known that the internal resonance and external force have a remarkable effect on the nonlinear vibration of the axially moving structures. Nayfeh and Balachandran [1] reviewed several different kinds of internal resonances and an external or parametric resonance for different dynamical systems. Balachandran and Nayfeh [2] studied the influence of modal interaction on the dynamic response of periodically forced composite structure through the experimental method. Riedel and Tan [3] investigated the coupled nonlinear vibration of axially moving strip, the response of the system is obtained with 3:1 internal resonance between the first two transverse modes. Here, we did not discuss the internal resonance for the axially moving hyperelastic beam.

There are numerous papers deal with parametric resonance of axially moving beam, string. Yang et al. [4] studied the nonlinear vibration behavior of axially moving strings based on gyroscopic modes decoupling. Özhan [5] studied the dynamics of the axially moving beams with variable speed and axial force, the stable and unstable regions for principal parametric resonances are obtained. Sahoo et al. [6,7] investigated the nonlinear transverse vibration of an axially moving beam subject to single and two frequencies excitation. The frequency response plots, amplitude curves, their stability and bifurcation are obtained by continuation algorithm. In [8], Chen applied the incremental harmonic balance method to study the stability and bifurcation of an axially moving beam with tuned to three-to-one internal resonance. Besides the linear material beams, there many researchers studied nonlinear dynamics behavior of axially moving viscoelastic beam. In [9], Ding studied the bifurcation and chaos of an axially acceleration viscoelastic beam in supercritical regime based on the high-order Galerkin method and differential and integral quadrature method. Chen and Tang [10] applied the multiple scales method to study the nonlinear vibration of an axially accelerating viscoelastic beam by taking into account the effects of longitudinally varying tensions. Tang et al. [11] studied the parametric and 3:1 internal resonances of axially moving viscoelastic beams on elastic foundation. The influence of the viscous damping coefficient on the dynamic behavior of the beam has been obtained by multiple scales method. In [12], Chen et al. investigated dynamic stability of an axially accelerating viscoelastic beam undergoing parametric resonance by Timoshenko thick beam theory. Recently, Ghayesh, Amabili and co-researchers have made significant investigations on...
traveling continua problems considering various aspects. In [13,14], Ghayesh and Amabili studied the nonlinear forced dynamics of an axially moving Timoshenko beam by taking into account rotary inertia and shear deformation in supercritical and subcritical regime. Ghayesh et al. [15-17] investigated the nonlinear forced vibration of axially moving beams in supercritical and subcritical regime with and without considering internal resonance. By taking into account the bending resistance and in-plane tension, Banichuk et al. [18] investigated the stability of axially moving plate with constant velocity by the analytical approach. Koivurova [19] studied the nonlinear vibration of axially moving membrane in three dimensional frame and the FEM numerical solutions have been used to testify the analytical results. In [20], Soares and Gonçalves studied the nonlinear vibration of pre-stress hyperelastic annular membrane under finite deformations, the vibration modes and frequencies of the membrane are obtained.

In order to improve the accuracy of the model equations, some researchers applied the high-order beam theory for the axially moving beam. Based on the Rayleigh beam theory, the authors applied finite element method to study the vibration and stability of an axially moving beam in [21]. By taking account into the shear and bulk effect, Seddighi and Eipacki [22] applied the multiple scales method to studied the dynamics response of an axially moving viscoelastic beam with time-dependent speed. Wang et al. [23] studied the nonlinear dynamical analysis of simply supported axially moving beam under finite deformation theory.

As mentioned above, the axially moving beam are composed of the linear or viscoelastic material. To the author’s knowledge, there are few papers deal with the nonlinear dynamic of axially moving hyperelastic beam. In this paper, hyperelasticity is integrated into axially moving material for the first time. Here, we consider not only the geometrical nonlinearity but also the material nonlinearity. Based on the continuum mechanics theory, the complicated nonlinear coupled model equations are obtained by using high-order shear deformation beam theory and Hamilton principle. In general, it is a hard task to obtained the analytical solutions of the coupled nonlinear governing equations. Then, the complex coupled equations are simplified into a single integro-differential equations about the transverse vibration based on the quasi-static stretch hypothesis. By introducing a small parameter, the leading order asymptotic analytical solutions of the principal parametric resonance are obtained through the multiple scales method. We also apply the Galerkin’s method to study the periodically vibration of the hyperelastic beam based on the coupled nonlinear model equations. For the amplitude, an excellent agreement between the model equations. Then, the complex coupled equations are simplified into the continuum mechanics theory. For this end, we consider the Lagrange strain $E$, which is used in calculations where large shape changes are expected. The expression of Lagrange strain is given by

$$E = \frac{1}{2} (F^T F - I),$$

where

$$F = \begin{pmatrix} 1 + u_1(x, y, t) & u_2(x, y, t) & 0 \\ u_2(x, y, t) & 1 + u_1(x, y, t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the deformation gradient tensor and $I$ is an identity second order tensor.

So, the nonlinear normal strain $\varepsilon_{xx}$ can be written as

$$\varepsilon_{xx} = u_{xx} + \frac{1}{2} u_{xx}^2 + \frac{1}{2} u_{xx}^2.$$

Substituting Eq. (1) into Eq. (4), we obtain

$$\varepsilon_{xx} = \varepsilon_{xx} = u_{xx} + \frac{1}{2} u_{xx}^2 + \frac{1}{2} u_{xx}^2 - y_1 (u_{xx} + (1 + u_{xx}) u_{xx} + (l + u_{xx}) u_{xx} + v_1 u_{xx} + v_2 u_{xx} + v_3 u_{xx} + O(y^3)).$$

In this paper, we only consider the second order moment of the area about the x-axis, then the higher order of $y$ have been omitted.

The virtual work done by the external tension $P$ is given by

$$\delta W = \int_A \int_0^L P \delta E_{xx} \, dA \, dx.$$

Substituting (5) into Eq. (6) and integrating the results over the cross section, we have

$$\delta W = \int_0^L P \delta ((u_{xx} + \frac{1}{2} u_{xx}^2 + \frac{1}{2} u_{xx}^2) + \frac{1}{2} v_1 u_{xx}^2 + u_{xx}^2 + v_1 u_{xx} + v_2 u_{xx} + v_3 u_{xx} + O(y^3)) \, dA.$$

where

$$I = \int_A y^2 dA, \quad \int_A y dA = 0.$$

are used.

The kinetic energy $T$ for the axially moving beam can be written as

$$T = \frac{1}{2} \rho \int_0^L \int_A \rho V \cdot V dA \, dx,$$

where

$$\rho V \cdot V dA = ((u_{xx} + y(t)(u_{xx})^2 + (u_{xx} + y(t)u_{xx})^2) \, dA.$$

Substituting (1) into Eq. (9) and integrating the results over the area, we have

$$T = \rho \int_0^L A(u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + u_{xx}^2 + v_1^2 u_{xx}^2 + v_2^2 u_{xx}^2 + v_3^2 u_{xx}^2)$$

In this study, the following in-plane kinematic frame [24] for the hyperelastic beam are adopted

$$\begin{cases}
    u_1(x, y, t) = u_0(x, t) - y u_{xx}(x, t), \\
    u_2(x, y, t) = u_1(x, t) - y v_1 u_{xx}(x, t) - \frac{1}{2} v_2 u_{xx}(x, t),
\end{cases}$$

where the parameter $v_1$ is the in-plane Poisson’s ratio of material, $u_0$ and $u_1$ are the longitudinal and transverse displacements of the beam at the neutral axis $y = 0$, respectively. Hereafter, the subscripts $x, t$ represent derivative with respect to $x, t$. For $v_1 = 0$, the formula is identical to the Euler beam theory where shear deformation is neglected.

Here, we will depict the deformation of hyperelastic beam based on the continuum mechanics theory. To this end, we consider the Lagrange strain $E$, which is used in calculations where large shape changes are expected. The expression of Lagrange strain is given by

$$E = \frac{1}{2} (F^T F - I),$$

where

$$F = \begin{pmatrix} 1 + u_1(x, y, t) & u_2(x, y, t) & 0 \\ u_2(x, y, t) & 1 + u_1(x, y, t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the deformation gradient tensor and $I$ is an identity second order tensor.

So, the nonlinear normal strain $\varepsilon_{xx}$ can be written as

$$\varepsilon_{xx} = u_{xx} + \frac{1}{2} u_{xx}^2 + \frac{1}{2} u_{xx}^2.$$
We expand the energy function. For the hyperelastic material, the strain energy function is very complex due to the material nonlinearities (nonlinear strain–stress relation). There are many kinds of hyperelastic material [25]. For simplicity, we assume the beam is composed of compressible Neo-Hookean material. For other hyperelastic material, we can deal with it by the same method. The strain energy function of Neo-Hookean material is given by [25]

\[ \Phi(I_1, I_3) = \mu \left( I_1 - \frac{3}{2} \right) - \frac{\nu}{2} (I_3 - 1)^2, \]

where \( \mu \) is the shear modulus, \( \nu \) is the bulk modulus as follows

\[ \mu = \frac{3E}{2(1+\nu)}, \quad \nu = \frac{E}{3(1-2\nu)} \]

\( v \) is the three dimensional Poisson’s Ratio and

\[ B = FF^T, \quad I_1 = Tr(B), \quad I_3 = Det(B). \]

Here, B is the left Cauchy–Green deformation tensor and \( I_1, I_3 \) are the invariant of B.

For the Neo-Hookean material, the in-plane Poisson’s ratio \( v_1 \) is given by

\[ v_1 = \frac{\nu}{1-\nu} \]

The total potential energy \( \phi \) of beam due to the deformation can be expressed as

\[ \phi = \int_0^L \Phi(I_1, I_3) \, dA. \]

Substituting Eq. (13) into Eq. (15), we have

\[ \phi = \int_0^L \int_0^L \frac{1}{2} \left( w_{xx} + w_{yx} + w_{xy} \right)^2 + \frac{1}{2} \left( \nu_1 + \nu_2 \right) \left( w_{xx}^2 + w_{yy}^2 - 2 \right) \right] \, dx \int_0^L \frac{1}{2} \left( w_{xx}^2 + w_{yy}^2 - 2 \right) \, dx \]

We expand the energy function \( \Phi \) for small strain the \( u_{xx} \) and \( u_{xy} \) up to the fourth order and integrating results over the area, we obtain the total potential energy \( \phi \). Here, the long expression is omitted.

Remark. In general, the longitudinal and the transverse displacements are very small. So, we keep the order of the transversal and the longitudinal displacements up to the third and the second order in model equations (18). Then some higher order nonlinear terms can be dropped from Eq. (16). Moreover, these higher order nonlinear terms will not influence the accuracy of the low-order asymptotic solutions of the model equations (18).

The extended Hamilton principle takes the form

\[ \int_{t_1}^{t_2} \left( \delta T - \delta F + \delta W \right) dt = 0, \]

According to the extended principle and after some manipulations, we obtain two coupled model equations as follows

\[ u_t + 2yu_x + (\gamma^2 - p - a_2)u_{xx} + \gamma(1 + u_x) \]

\[ = a_4 u_{xx} \]

\[ w_{xx} + 2w_{xx} + (\gamma^2 - p - a_2)w_{xx} + \gamma (1 + u_x) \]

\[ = a_4 w_{xx} \]

\[ w_{xx} + 2w_{xx} + (\gamma^2 - p - a_2)w_{xx} + \gamma (1 + u_x) \]

\[ = a_4 w_{xx} \]

\[ \int_{t_1}^{t_2} \left( \delta T - \delta F + \delta W \right) dt = 0, \]

with the following dimensionless parameters:

\[ x = Lx^*, \quad u_0 = Lu^*, \quad w_0 = Lu^*, \quad t = \sqrt{\frac{E}{pL^2}}, \quad \gamma = \sqrt{\frac{E}{p}} \gamma^*. \]

\[ p^* = \frac{P}{AL^2}, \quad k_{ij} = \frac{I_{ij}}{AL^2}. \]

For convenience, the superscript * is removed. Here \( a_i (i = 1, 2, \ldots, 9) \) are constants depending on the material parameters \( p, E \). For a beam with rectangular cross-section, the geometrical parameter \( k_{ij} \) can be written as

\[ k_{ij} = \frac{I_{ij}}{AL^2} = \frac{1}{I} d^2 \]

where \( \delta \) is the width of the cross-section of the beam. It means that the parameter \( k_{ij} \) is very small for a slender beam.

The following specific boundary conditions have to be satisfied for a hinged–hinged beam with immovable edges:

\[ \begin{cases} 
  u = 0, & at \quad x = 0, \\
  w = 0, & at \quad x = 0, 
\end{cases} \]

The model equations (18) are very complex, it is very hard to obtain the analytical solution. Wickert [26] assume the quasi-static stretch hypothesis to establish an integro-partial-differential equation for the transverse motion. Here, we applied the same assumption for the axially accelerating beam. Furthermore, in general, the geometrical parameter \( k_{ij} \) is very small, then we can omit the relevant terms. So, the (18) can be re-write as

\[ a_4 u_{xx} + \gamma w_{xx} + (p + a_2)u_{xx} = 0, \]

According to Eq. (22) and the boundary conditions, we have

\[ u(x, t) = \int_0^L u(x) \, dx = \frac{a_4}{2(p + a_2)} \int_0^L u(x)^2 \, dx. \]

Now, substituting (23) into (18), it lead to

\[ u_{tt} + 2u_{tt} + \gamma u_{tt} - (p)^2 u_{xx} + (1 - \nu_1) k_{ij} u_{xx} + w_{xx} + 2\gamma w_{xx} \]

\[ = a_4 u_{xx} \]

\[ k_{ij} = \frac{1}{4(p + a_2)^2} \int_0^L u_{xx}^2 \, dx \]

\[ a_4 = \frac{a_4}{2(p + a_2)^2} \int_0^L \gamma u_{xx} \, dx. \]

\[ a_4 = \frac{a_4}{2(p + a_2)^2} \int_0^L \gamma u_{xx} \, dx. \]

In next sections, we will studied the nonlinear vibration of the axially moving hyperelastic beam through the model equation (24) and boundary condition (21).

3. Analysis of principal parametric resonance

3.1. Multi-scale analysis

Assuming that the velocity is harmonically varying about a constant mean velocity \( \gamma_0 \), one writes

\[ \gamma(t) = \gamma_0 + \gamma_f \sin(\Omega t). \]

where \( \gamma_f \) presents the amplitude of the fluctuations, which is small, \( \omega \) is the fluctuations frequency.

For the perturbation procedure, we let

\[ u(x, t) = \sqrt{\varepsilon} u(x, t). \]

According to multiple scales method, we let

\[ u(x, t) = u(x, t) + \varepsilon^2 u(x, t) + \varepsilon^4 u(x, t) + \cdots, \]

where \( T_1 = \varepsilon T_0 \).
Substituting (26)–(28) into (24) and equating the coefficient of each power of e to zero, we have a system of equations. Especially, from the coefficients of \(e^i, e^1\), we have two sets of equations as following:

\[
\begin{align*}
&h_{0TT} + 2h_{0X}Y_0 + h_{0XX}(r_0^2 - p) - (1 - v_1)k_1h_{0XXTT} - 2k_1Y_0(1 - v_1)h_{0XXXT} - k_1[(v_1 - 1) - a_1 + r_0^2(1 - v_1)]h_{0XXX} = 0, \\
&-h_{1TT} + 2h_{1X}Y_0 + 2(1 - v_1)k_1h_{1XXTT} - h_{1XXX}(r_0^2 - p) + (1 - v_1)k_1h_{1XXX} + k_1h_{1XXXT}(v_1 - 1) + 2a_1 + (1 - v_1)Y_0 - 2h_{0T}, \\
&-2h_{0X}(1 + a_1 + 2k_1h_{0XXT} + (2 - 2v_1)k_1h_{0XXX} + h_{0XX}(1 - v_1)cos(T\Omega)k_1Y_1h_{0XX}, \\
&+ h_{0XX}(1 - v_1)cos(T\Omega)k_1Y_1h_{1XX}, + (a_1^2 k_1^2 h_{0XX}^2 dx)/(2(p + a_1)) \\
&- 2 \sin(T\Omega)Y_0Y_1 + a_1h_{0XX}^2h_{0T} + k_1(a_1^2 k_1 h_{0XX}^2 dx) + (2 - 2v_1)\sin(T\Omega)Y_0Y_1 + a_1h_{0XX}^2h_{0T} = 0.
\end{align*}
\]

(29)

The general solution of (29) can be write as:

\[h_0(x, T_0, T_1) = Y_0(x)A_0(T_1)e^{-i\omega_0T_0} + Y_1(x)A_1(T_1)e^{-i\omega_1T_0},\]

where \(\omega_0\) is the nth natural frequency of the transverse vibration, \(Y_0(x)\) and \(A_0(T)\) are the nth complex modal function and the amplitude of modal function, respectively. Substituting (31) into Eq. (29), we obtain:

\[
\begin{align*}
&\frac{a_1^2 k_1^2 h_{0XX}^2 dx}{2(p + a_1)} \\
&+ a_1h_{0XX}^2h_{0T} + k_1(a_1^2 k_1 h_{0XX}^2 dx).
\end{align*}
\]

(30)

It is easy to know that Eq. (31) have four different roots \(p_{ij}(j = 1, 2, 3, 4)\). Then, the solution of (32) is:

\[
\phi(x) = C_a e^{i\beta_1x} + C_b e^{i\beta_2x} + C_c e^{i\beta_3x} + C_d e^{i\beta_4x}
\]

(34)

where \(C_a\) are constants to be determined. Substituting (34) into the boundary conditions equation (21), a linear algebraic system for \(C_a\) are obtained. Then, the model function can be written as:

\[
\begin{align*}
&\phi_a(x) = C_1 e^{i\beta_1x} = (\beta_1^2 - \beta_2^2)\psi_0(x) - \beta_1 \psi_1(x) \\
&+ \beta_1^2 \psi_2(x) + \beta_1 \psi_3(x) e^{-i\beta_1x}, \\
&\psi_0(x) = (\beta_1^2 - \beta_2^2)\psi_0(x) - \beta_1 \psi_1(x) \\
&+ \beta_1^2 \psi_2(x) + \beta_1 \psi_3(x) e^{-i\beta_1x}, \\
&\psi_1(x) = (\beta_2^2 - \beta_3^2)\psi_0(x) - \beta_2 \psi_1(x) \\
&+ \beta_2^2 \psi_2(x) + \beta_2 \psi_3(x) e^{-i\beta_1x}, \\
&\psi_2(x) = (\beta_3^2 - \beta_4^2)\psi_0(x) - \beta_3 \psi_1(x) \\
&+ \beta_3^2 \psi_2(x) + \beta_3 \psi_3(x) e^{-i\beta_1x}.
\end{align*}
\]

(35)

For non-trivial solutions, the determinant of the coefficient matrix must be zero. Then, we have:

\[
\begin{align*}
&(\beta_1^2 - \beta_2^2)\psi_0(x) - \beta_1 \psi_1(x) + (\beta_2^2 - \beta_3^2)\psi_0(x) - \beta_2 \psi_1(x) + (\beta_3^2 - \beta_4^2)\psi_0(x) - \beta_3 \psi_1(x) = 0.
\end{align*}
\]

(36)

\(\omega_0\) and \(\beta_0\) can be solved by numerical method from Eqs. (36) and (33). Substituting (31) into (30), we have:

\[
\begin{align*}
&h_{1TT} + 2h_{1X}Y_0 + h_{1XX}(r_0^2 - p) - (1 - v_1)k_1h_{1XXTT} - 2k_1Y_0(1 - v_1)h_{1XXXT} - k_1[(v_1 - 1) - a_1 + r_0^2(1 - v_1)]h_{1XXX} = 0, \\
&-h_{0T} + 2h_{0X}Y_0 + 2(1 - v_1)k_1h_{0XXTT} - h_{0XXX}(r_0^2 - p) + (1 - v_1)k_1h_{0XXX} + k_1h_{0XXXT}(v_1 - 1) + 2a_1 + (1 - v_1)Y_0 - 2h_{0T}, \\
&-2h_{0X}(1 + a_1 + 2k_1h_{0XXT} + (2 - 2v_1)k_1h_{0XXX} + h_{0XX}(1 - v_1)cos(T\Omega)k_1Y_1h_{0XX}, \\
&+ h_{0XX}(1 - v_1)cos(T\Omega)k_1Y_1h_{1XX}, + (a_1^2 k_1^2 h_{0XX}^2 dx)/(2(p + a_1)) \\
&- 2 \sin(T\Omega)Y_0Y_1 + a_1h_{0XX}^2h_{0T} + k_1(a_1^2 k_1 h_{0XX}^2 dx) + (2 - 2v_1)\sin(T\Omega)Y_0Y_1 + a_1h_{0XX}^2h_{0T} = 0.
\end{align*}
\]

(37)

where C.C is the complex conjugate of the previous two terms and:

\[
\tau_j(x) = \int_0^x \omega_j^2 ds, \quad \tau_j(x) = \int_0^x Y_j^2 ds, \quad \tau_j(x) = \int_0^x Y_j^2 ds
\]

(38)

A detuning parameter \(\xi\) is introduced to quantify the deviation of \(\Omega_\epsilon\), it is given by:

\[
\Omega = 2\omega_\epsilon + \xi.
\]

(39)

According to the solvable condition, we have:

\[
\frac{dA_0}{dT_1} + \gamma_1 f_1 A_1 e^{i\tau_1 T_1} + f_2 A_2 e^{i\tau_2 T_1} = 0,
\]

(40)

where \(f_1\) and \(f_2\) are given by:

\[
\begin{align*}
&f_1 = \int_0^1 \left( \int_0^{\tau_1 T_1} - \frac{1}{2} \Omega - \nu_0 Y_{nx} + \left( \frac{1}{2} (1 - v_1) \Omega k_1 \right) + \int_0^{\tau_1 T_1} \nu_0 Y_{nx} \right) dx, \\
&f_2 = \int_0^1 \left( \nu_0 Y_{nx} \left( \frac{2}{p + a_1} \right) + \nu_0 Y_{nx} \left( \frac{2}{p + a_1} \right) \right) dx, \\
&\gamma_1 = \int_0^1 \left( \frac{a_1 \nu_0 k_1}{4(p + a_1)} \right) dx.
\end{align*}
\]

(41)

(42)

Substituting (42) into (40) and separating the real and imaginary part of resulting, we obtained two equations as following:

\[
\begin{align*}
&\rho_{0T_1} = f_1 \rho_0 + f_2 \rho_0 q_0 + f_1 \rho_0 + \frac{\eta_0}{2} - r_1 f_1 \rho_0 - f_2 \rho_0 - r_1 f_1 \rho_0, \\
&\rho_{T_1} = r_1 f_1 \rho_0 - f_2 \rho_0 q_0 - f_2 \rho_0 - r_1 f_1 \rho_0 - \frac{\eta_0}{2} - f_2 \rho_0.
\end{align*}
\]

(43)

At \(\rho_0 = q_0 = 0\), the Jacobian matrix is given by:

\[
\begin{align*}
&|J - \lambda I| = -r_1 f_1 - \frac{1}{2} \lambda - r_1 f_1 - 1 - \frac{1}{2} \lambda - r_1 f_1 - \frac{1}{2} \lambda, \\
&-r_1 f_1.
\end{align*}
\]

(44)

Then, the characteristic equations takes the form:

\[
\lambda^2 - \left( r_1 f_1 \right)^2 = \lambda^2 - \frac{\eta_0^2}{4} = 0.
\]

(45)

when:

\[
-2r_1 f_1 |f_1| < \sigma < 2r_1 |f_1|.
\]

(46)
the eigenvalues of Eq. (45) have the positive real parts. According to the Lyapunov stability theory, the unstable region of the trivial solution is given by (46).

In order to study the stability boundaries of the nontrivial solution, we assume Eq. (40) has a solution for $A_n(T_i)$ in polar form

$$A_n(T_i) = a_n(T_i) e^{i \theta_n(T_i)}$$

where $a_n$ and $\theta_n$ are real value functions of $T_i$. Substituting (47) into (40) and separating the resulting equations into real and imaginary parts, we obtain two equations as following

$$\frac{da_n}{dT_i} = a_n (\gamma_1 f_1' \sin(\eta) - \gamma_1 f_1'' \cos(\eta)) - f_2^R \alpha_n,$$

$$\frac{d\eta}{dT_i} = 2 (\gamma_1 f_1' \sin(\eta) + f_1'' \cos(\eta)) + \sigma + 2 f_2^R \alpha_n,$$

where

$$\eta(T_i) = \sigma T_i - 2 \beta_s(T_i)$$

Then, the steady state response of the system can be obtained as following

$$4 |f_2|^2 a_n^2 + 4 f_2^R \sigma a_n^2 - 4 |\gamma_1 f_1|^2 + \sigma^2 = 0.$$  

(50)

From above equations, we have

$$\sigma = 2 (-f_2^R a_n^2 \pm \sqrt{|\gamma_1 f_1|^2 - (f_2^R)^2 a_n^2}).$$  

(51)

Constructing the Jacobian matrix from modulation equations and evaluating the eigenvalues lead to

$$|J - \lambda I| = \begin{vmatrix} -2 f_2^R a_n^2 - \lambda - \frac{1}{4} a_n (2 f_2^R a_n^2 + \sigma) & 4 f_2^R a_n \\ 4 f_2^R a_n & -2 f_2^R a_n^2 - \lambda \end{vmatrix}$$

(52)

directly calculation of the determinant leads to the characteristic equation as

$$\lambda^2 + 4 f_2^R a_n^2 \lambda + 4 (f_2^R)^2 a_n^2 \pm 4 f_2^R a_n^2 \sqrt{|\gamma_1 f_1|^2 - (f_2^R)^2 a_n^2} = 0.$$  

(53)

when

$$f_2^R > 0. \quad 4 (f_2^R)^2 a_n^2 \pm 4 f_2^R a_n^2 \sqrt{|\gamma_1 f_1|^2 - (f_2^R)^2 a_n^2} > 0.$$  

(54)

the real parts of the eigenvalues of Eq. (53) are negative number. Then, the stability condition of the nontrivial solution for the detuning parameter is given by (54) based on the Lyapunov stability theory.

### 3.2. Results and discussions

In this subsection, the equilibrium and dynamic behavior of the hyperelastic beam traveling with time-dependent velocity and subject to principle parametric resonance is carried out by the numerical experiments based on Eq. (50). For given the parameters, the frequency response curve are plotted to highlight their effect on the resonant response.
Firstly, we research the influence of the in-plane Poisson’s ratio \( \nu_1 \) on the nonlinear vibration of the hyperelastic beam. To this end, we fixed \( k_i = 0.0004, \rho = 0.0004 \) and \( \gamma_0 = 0.04 \), the frequency response curves for the first model is plotted in Fig. 1(a). Hereafter, the black dotted line and solid blue line represent the unstable and stable equilibrium solutions, respectively. It shows that the type of non-linearity of the hyperelastic beam is softening when the in-plane Poisson’s ratio is very small. However, the type of non-linearity is hardening for \( \nu_1 = 0.5, 0.75 \). In other words, increasing \( \nu_1 \) cause the frequency response to bend more to right and hence the hardening behavior of the beam increase. In the hardening case, the amplitude of vibration will decrease with the increasing of the in-plane Poisson’s ratio \( \nu_1 \). For \( \nu_1 = 0 \) and \( \nu_1 = 0.25 \), the nontrivial steady state solutions are stable when the detuning parameter \( \sigma \) increase accordingly until the sub-bifurcation points at \( \sigma = 0.003624 \) and \( \sigma = 0.003543 \) are reached, respectively. The sub-bifurcation will occur when the detuning parameter \( \sigma \) increase for \( \nu_1 = 0.5, 0.75 \). In this case, the solution branch from the bifurcation points is unstable. Furthermore, the numerical results shows that the stable range of the trivial solutions become large with the growth of the in-plane Poisson’s ratio \( \nu_1 \).

According to (12) and (14), the ratio of bulk modulus \( \beta \) to shear modulus of \( \mu \) of the hyperelastic material can be written as

\[
\frac{\beta}{\mu} = \frac{2(1+2\nu_1)}{3(1-\nu_1)} \quad (55)
\]

Then, for the effect of \( \beta/\mu \) on the nonlinear vibration, we have the similar results with to parameter \( \nu_1 \). It means that the effect of the characteristic of hyperelastic material on the nonlinear vibration can be qualitatively analyzed by (55).

In Fig. 1(b), the influence of the geometric parameter \( k_i \) on the frequency response is shown. Here, we only show the unstable interval of the trivial solutions for \( k_i = 0.0012 \) in this figure, the unstable interval is given by \((-0.018774, 0.018774)\). We find that the amplitude of the system decrease with the decreasing of the geometrical parameter \( k_i \). The unstable interval of trivial solution become small with the increasing of \( k_i \), which means the thick beam become more stable during the axially moving.

In Fig. 2(a), we find when \( \gamma_0 \) increase from 0.02 to 0.04, the amplitude of nontrivial solutions of the system become large. However, the amplitude become very small for \( \gamma_0 = 0.06 \), in this case, the speed is very closed to the critical speed \( \gamma_c = 0.0684734 \). After some numerical experiments, the results show that the amplitude may dramatically decrease when the velocity \( \gamma_0 \) is very closed to the critical velocity. In general, the mean axial velocity \( \gamma_0 \) and the vibration amplitude are positively correlated for linear material beam, which is in contrast with our results according to Fig. 2(a). It means that the nonlinear dynamical of the hyperelastic beam is very complex. In Fig. 2(b), the effect of parameter \( \gamma_1 \) on the frequency response is presented. The stable range of the trivial solutions and the amplitude of nontrivial solutions of the steady state are increase with the growth of the \( \gamma_1 \).

4. Numerics via the Galerkin method

In this section, we apply the Galerkin method to solve the coupled nonlinear governing equations (18). The longitudinal and transverse motion are approximated by the following form:

\[
\begin{align*}
\bar{u}(x, t) &= \sum_{i=1}^{N} \phi_i(t) \phi_i(x), \\
\bar{t}(x, t) &= \sum_{i=1}^{N} p_i(t) \phi_i(x),
\end{align*}
\]

(56)

where \( \phi_i(x) = \sin(i\pi x) \) represents the \( i \)th eigenfunction for the transverse motion of a simple support stationary hyperelastic beam and \( p_i(t), q_i(t) \) denote the \( i \)th generalized coordinate for the transverse and the longitudinal motions, respectively.

Substituting Eq. (56) into (18) and applying Galerkin’s method results in a set of coupled nonlinear second-order ODEs as following:

\[
\begin{align*}
\sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) q_n + 2\gamma \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) q_i \\
+ (r^2 - a_0 - \rho) \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) q_i + \gamma \int_0^1 \phi_i dx \\
+ \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) q_i - v_i^2 k_i \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) q_i \\
= a_0 \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) p_i + p_0 \\
+ k_i \left[ \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) p_i \right] \\
+ a_0 \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) p_i + v_i^2 \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) q_i \\
+ 2v_i^2 \gamma \sum_{i=1}^{N} \left( \int_0^1 \phi_i \phi_j dx \right) a_i, \quad j = 1, \ldots, N
\end{align*}
\]

(57)

Above system equations can be solved numerically using direct time integration by the variable step-size Runge–Kutta method. To this end, we fix \( N = 4 \) and the initial conditions are given by

\[
\begin{align*}
q_1(0) &= 0.00001 \sin(x), \quad q_2(0) = 0, \quad q_3(0) = 0, \\
q_4(0) &= 0, \quad n = 2, 3, 4 \\
p_1(0) &= 0.001 \sin(x), \quad p_2(0) = 0, \quad p_3(0) = 0, \\
p_4(0) &= 0, \quad n = 2, 3, 4
\end{align*}
\]

(58)

In numerical calculation, the amplitude \( A_i \) of the stable steady-state response is defined by the largest displacement of the beam center. It should be noted that the largest displacement of the beam is independent the initial boundary conditions (58).

In this section, we only consider the effect of \( \nu_1 \) on the frequency response and the type of nonlinear vibration behavior of the axially moving hyperelastic beam. The system parameter are fixed as \( k_i = 0.0004, \gamma_0 = 0.4 \) and \( \gamma_1 = 0.0016 \). Numerical results show that the beam moving periodically in Fig. 3 with system parameter \( \nu_1 = 0 \). In this case the first natural frequency \( a_1 = 0.157644 \). For \( \nu_1 = 0.5 \), then the
first natural frequency $\omega_1 = 0.172328$. The time history of the dynamic behavior are plotted in Fig. 4. In Fig. 5, the phase portraits are plotted for $\nu_1 = 0$.

Now, the effect of the in-plane Poisson’s ratio $\nu_1$ on the stable steady-state oscillating responses are investigated by numerical results based on Eq. (58). Meantime, the comparison between the analytical results and the numerical results of stable steady-state oscillating responses are made.

According to (31) and (34), the leading order asymptotic analytical solution can be written as

$$
u(0.5, t) = A_1 \left( \exp\left(i(\omega_1 t + \sigma)\phi(0.5) \right) + \exp\left(-i(\omega_1 t + \sigma)\phi(0.5) \right) \right)$$

(59)

The maximal value of (59) is the stable steady-state oscillating responses.
For the given parameters, the comparison between the analytical results and the numerical results are plotted in Fig. 6. In this figure, the solid red lines and the dotted blue lines denote the results of the multiple scales and the Runge–Kutta method, respectively. It shows that the analytical results are very good approximate the numerical results. The difference between the numerical and analytical solution become large with the decreasing of $\sigma$ for the $\nu_1 = 0$. One reason is that the analytical solution of multiple scales is the leading order while in the Galerkin method the fourth order asymptotic solutions are presented. For $\nu_1 = 0.5$, we have the similar conclusion. Furthermore, for the effect of the $\nu_1$ on the steady-state response and the type of nonlinear vibration behavior, the results of multiple scales method are consist with that of the numerical method.

5. Conclusions

The nonlinear vibration of axially accelerating hyperelastic beam with the simple support boundary condition has been investigated in this paper. By multiple scales method, the effect of the system parameters on the nonlinear vibration have been investigated based on an integro-differential equation. The results obtained in this paper are given in the following:

1. For the small value of in plane Poisson’s ratio, the type of the nonlinear vibration behavior is softening, however, the type become hardening for the large value of the in plane Poisson’s ratio. The effect of characteristic hyperelastic material $r_c$ can be determined through (55). In other words, the small bulk modulus will lead to the appearance of softening type of the nonlinear behavior. The type of the nonlinear behavior become hardening for the large bulk modulus. Furthermore, the amplitude of the frequency response will decrease with the growth of bulk modulus. These results have not presented in existing literature.

2. The amplitude of the hyperelastic beam decrease with the decreasing of the geometric material.

3. The numerical results show that the amplitude of the frequency response may dramatically decrease when the mean velocity $\gamma_0$ is very closed to the critical speed of the beam. The stable range of the trivial solutions and amplitude of nontrivial solutions of the steady state are increasing with the growth of the $\gamma_1$.

Lastly, in order to verify the analytical results, the Galerkin method are applied to solve the coupled nonlinear partial differential model equations to obtain the periodic vibration behavior. It shows that the results of multiple scales method are consist with that of the Galerkin method.

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