Nonlinear isolation of transverse vibration of pre-pressure beams

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Research on nonlinear isolation has always focused on the vibration of discrete systems. The elastic vibration of the main structure itself is always ignored. In order to study the influence of the multimodal elastic vibration of the main structure on the nonlinear isolation effect, nonlinear isolation of the transverse vibration of a pre-pressure elastic beam is studied in this paper. Three linear springs are utilized to build a nonlinear isolation system, in which horizontal springs are utilized to provide non-linearity and to achieve quasi-zero stiffness. The transverse vibration of the beam is isolated by the elastic support at the two ends. The Galerkin truncation method (GTM) is used to solve the response of the forced vibration. The isolation effect of the primary resonance of the pre-pressure beam is presented. The effects of the axial pre-pressure and nonlinear isolation system on vibration transmission are explored. Results of the GTM are confirmed by utilizing the finite difference method (FDM). It also illustrates the effectiveness of the proposed difference method for nonlinear support structures. The numerical results demonstrate that under certain conditions, the quasi-zero stiffness isolation system may increase the transmission of high-order modal vibration of the elastic continuum. Furthermore, this work finds that the initial axial pre-pressure could be beneficial to vibration isolation of elastic structures.

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1. Introduction

The objective of this paper is to study nonlinear isolation of the transverse vibration of an axial pre-pressure beam. The unexpected vibration of elastic structures is a challenging problem [1–3]. Moreover, the vibration hinders the development of modern engineering and advanced technology. Therefore, the study on vibration of elastic structure has continued for centuries and is still a hot research issue.

The study of the vibration characteristics of elastic structures usually assumes that the boundaries of the structure are simply supported, fixed, or completely free [4]. Only a few studies have considered the imperfections of the boundary conditions. Random response of a uniform beam restrained at each end by a rotational and a translational spring was considered [5]. The complex modes and the forced vibration of a beam with viscoelastic boundary supports were studied [6]. The orthogonality condition for the Eigen functions of a non-uniform Timoshenko beam [7] and a non-uniform plates [8] with...
general time-dependent boundary conditions was derived. Vibration characteristics of elastically restrained beams attracted some attention [9–12]. The stability and natural characteristics of a supported beam with rotational spring constraints were analyzed [13]. The vibration of dynamic elastic structures with elastic boundary conditions is also of concern. A rotating Rayleigh beam [14] and a straight fluid-conveying pipe [15] considering the gyroscopic effect were, respectively, presented with arbitrary boundary conditions, specified in terms of elastic constraints in the translations and rotations, or even in terms of attached lumped masses and inertias. Although some attention has been paid to the vibration of elastic structures with general boundary conditions, in the above-mentioned literature, the boundary conditions were all assumed to be linear. The significant influence of nonlinear boundaries on the vibration of flexible structures was explored [16,17]. However, the influence of the nonlinear elastic support on the vibration transmission of the elastic structures has not been paid much attention.

Unexpected vibrations of the elastic structures can cause harmful effects [18–20]. Research on the vibration control of elastic structures has been widely concerned [21–24]. Vibration isolators have significant advantages, including simplicity and no additional energy. Therefore, passive vibration isolation is still the preferred solution to the vibration control problem in engineering [25,26]. Nonlinear vibration isolation can effectively overcome the shortcomings of linear isolation in the isolation bandwidth and stability. Considerable progress has been achieved in the study of nonlinear vibration in discrete systems. Based on a large amount of literature, Ibrahim reviewed the developments of nonlinear isolators of the vibration before 2008 [27]. The effects of stiffness [28–30] and nonlinear viscous damping [31–34] on vibration isolation were presented. Nonlinear vibration isolation via a scissor-like structured platform was investigated [35,36]. Vibration isolation platform was designed for a whole-spacecraft system [37]. Optimally passive vibration isolation system was developed for a single degree-of-freedom system [38]. A vibration isolator with high damping was designed and fabricated [39]. The vibration transmissibility of an oscillator with nonlinear fractional order damping was investigated [40]. Advances in micro-vibration isolation were reviewed before 2015 [41].

In order to overcome the restriction between the isolation frequency range and the load bearing capacity of linear isolators, many nonlinear quasi-zero stiffness mechanisms have been proposed [42–44]. To counteract the positive stiffness of a rubber spring, a torsion magnetic spring was employed to produce negative torsion stiffness [45]. Negative stiffness vibration isolation systems have been proved to offer a unique passive approach for achieving low vibration environments [46–48]. The starting frequency of isolation of the nonlinear isolator by using Euler buckled beam as negative stiffness elements were found to be lower than that of the linear one with the same support capacity [49,50]. A negative stiffness system based on a set of two double-acting pneumatic linear actuators was developed [51]. Stochastic resonance in mechanical vibration isolation system consists of a vertical linear spring in parallel with two horizontal springs was studied [52]. With geometric nonlinear damping, the transmissibility of a quasi-zero stiffness vibration isolator is proposed [53]. The study of the transmissibility of high dimensional quasi-zero stiffness system has also received attention [54–58]. These studies show that quasi-zero stiffness systems have the low force transmissibility and the low frequency band for vibration isolation. However, in all of the above-mentioned studies related to nonlinear vibration isolation, the elastic vibration of the isolated main structure has been neglected.

In this paper, a pre-pressure beam with nonlinear isolation system is established. Three linear springs are utilized to construct a nonlinear isolation system with quasi-zero stiffness. The Galerkin method is developed to solve the response and the vibration transmission of the primary resonance. Moreover, the finite difference format is proposed to solve the partial differential equation with nonlinear boundaries. Both methods are mutually verified. The influence of the quasi-zero stiffness system on isolation of multimodal vibration of elastic structures is firstly presented.

2. Dynamic model

The mechanical model of transverse vibration of a pre-pressure beam with nonlinear support is shown in Fig. 1. \( L \) represents the length of the beam. At both boundaries of the beam, three springs combine into a quasi-zero stiffness support system.
condition. \( T \) and \( X \) are, respectively, the time and the axial coordinate of the beam. \( K_{\text{vl}} \) and \( K_{\text{vr}} \) are, respectively, the stiffness of the vertical springs at the left end and right end. In this work, the horizontal linear spring \( K_{H} \) and the support damping \( C_{S} \) at the two boundaries are the same respectively. \( L_{0} \) is the length of the horizontal linear springs when they are in the horizontal position. \( P_{0} \) describes the axial pre-pressure. The pre-pressure beam is subjected to a uniformly distributed force \( f(X, T) = F_{0}\cos(\Omega T) \), where \( F_{0} \) and \( \Omega \) are, respectively, the magnitude of the linear density and the frequency of the external force. \( U(X, T) \) presents the transverse vibration displacement. \( L_{0} \) is the initial length of the horizontal linear spring. In the vertical direction at the pre-pressure beam boundaries, the relationship between the force and the displacement produced can be expressed as the following equation [52].

\[
X = 0 : F_{\text{BL}} = K_{\text{VL}} U + C_{S} \frac{\partial U}{\partial T} + 2K_{H} \left(1 - \frac{L_{0}}{\sqrt{U^{2} + L_{H}^{2}}} \right) U, \\
X = L : F_{\text{BR}} = K_{\text{VR}} U + C_{S} \frac{\partial U}{\partial T} + 2K_{H} \left(1 - \frac{L_{0}}{\sqrt{U^{2} + L_{H}^{2}}} \right) U
\]

(1)

\( F_{\text{BL}} \) and \( F_{\text{BR}} \) are the vertical support forces at both boundaries. For \( U(0, T) < 0.2L_{H} \) and \( U(L, T) < 0.2L_{H} \), the relationship can be approximated by

\[
X = 0 : F_{\text{BL}} = K_{L} U + K_{N} U^{3} + C_{S} \frac{\partial U}{\partial T}, \\
X = L : F_{\text{BR}} = K_{R} U + K_{N} U^{3} + C_{S} \frac{\partial U}{\partial T}
\]

(2)

where

\[
K_{L} = K_{\text{VL}} - 2K_{H} \left(\frac{L_{0}}{L_{H}} - 1\right), K_{R} = K_{\text{VR}} - 2K_{H} \left(\frac{L_{0}}{L_{H}} - 1\right), \\
K_{N} = K_{H}L_{0}/L_{H}^{3}
\]

(3)

The geometric nonlinearity of the transverse deformation of the pre-pressure beam is accounted for by utilizing the Lagrange strain. Therefore, the following geometric nonlinearity is introduced based on the quasi-static stretch assumption

\[
P = \frac{1}{l} \int_{0}^{L} \left[ 1 - \frac{2EA}{l} \left( \frac{\partial U}{\partial X} \right)^{2} \right] dX
\]

(4)

The damping force and the nonlinear restoring force of the supporting springs at the boundaries are regarded as the concentrated load of the pre-pressure beam. Based on the Newton’s second law of motion, the nonlinear integral-partial differential equation is established as

\[
\rho A \frac{\partial^{2} U}{\partial T^{2}} + EI \frac{\partial^{4} U}{\partial X^{4}} + C_{B} \frac{\partial U}{\partial T} + P_{0} \frac{\partial^{2} U}{\partial X^{2}} + F_{0} \cos(\Omega T) \\
+ \delta(X) \left( C_{S} \frac{\partial U}{\partial T} + K_{N} U^{3} \right) + \delta(X - L) \left( C_{S} \frac{\partial U}{\partial T} + K_{N} U^{3} \right) = \frac{EA}{2l} \frac{\partial^{2} U}{\partial X^{2}} \int_{0}^{L} \left( \frac{\partial U}{\partial X} \right)^{2} dX
\]

(5)

where \( \rho \) and \( A \) are, respectively, the density and the cross-sectional area of the uniform pre-pressure beam. \( I \) and \( E \) respectively describe the moment of inertia and the elastic modulus. \( C_{B} \) is the external viscous damping coefficient, introduced for concerning the energy dissipation in the transverse vibration of the pre-pressure beam. Then the linear elastic boundary conditions of the pre-pressure beam can be expressed as

\[
X = 0 : K_{L} U + EI \frac{\partial^{2} U}{\partial X^{2}} + P_{0} \frac{\partial U}{\partial X} = 0, \quad \frac{\partial^{2} U}{\partial X^{2}} = 0, \\
X = L : K_{R} U - EI \frac{\partial^{2} U}{\partial X^{2}} - P_{0} \frac{\partial U}{\partial X} = 0, \quad \frac{\partial^{2} U}{\partial X^{2}} = 0
\]

(6)
In the steady-state response phase, the force transmissibility of the transverse vibration at the two ends of the pre-pressure beam can be defined as

\[
X = 0 : \eta_L = \frac{K_L U + C \frac{dU}{dT} + K_N U^3}{P_0 L/2},
\]

\[
X = L : \eta_R = \frac{K_R U + C \frac{dU}{dT} + K_N U^3}{P_0 L/2}.
\]

By defining the following dimensionless parameters and variables

\[
u = \frac{U}{L}, x = \frac{x}{L}, t = \sqrt{\frac{P_0}{\rho A L^2}}, c_B = \frac{L^2}{P_0 A P_0}, k_1 = \frac{E A}{P_0}, k_f = \sqrt{\frac{E I}{P_0 L^2}}, f = \frac{F_0 L}{P_0}, k_L = \frac{L K_k}{k_f}, k_R = \frac{L K_k}{k_f}, k_{NL} = \frac{L^3 K_{NL}}{P_0}, k_{NR} = \frac{L^3 K_{NR}}{P_0}, \]

Then the governing equation and boundary conditions can be expressed as dimensionless

\[
u_{xx} + c_B u_{tt} + u_{xx} + k_1^2 u_{xxxx} + f \cos(\omega_0 t) - \frac{k_1}{2} u_{xx} \int_0^1 u_x^2 dx = \]

\[-\delta(x) \left( c_1 u_{tt} + k_{NL} u^3 \right) - \delta(x-1) \left( c_R u_{tt} + k_{NR} u^3 \right) \]

\[k_L u(0, t) + u_{xxx}(0, t) + \frac{u_x(0, t)}{k_f} = 0, \quad k_R u(1, t) - u_{xxx}(1, t) - \frac{u_x(1, t)}{k_f} = 0, \]

\[u_{xx}(0, t) = 0, \quad u_{xx}(1, t) = 0. \]

Since the numerical calculation of the dimensional equation is faster, in this paper, the dimensional equation is used for the actual calculation.

3. Linear free vibration of the pre-pressure beam

Without excitation and damping, the corresponding equation of the linear transverse vibration of the pre-pressure beam can be written as

\[
\rho A \frac{\partial^2 U}{\partial T^2} + E I \frac{\partial^2 U}{\partial X^4} + P_0 \frac{\partial^2 U}{\partial X^2} = 0
\]

The transverse vibration displacement can be assumed as

\[U(X, T) = \phi(X) e^{i \omega T} \]

where \( \phi(X) \) and \( \omega \) are, respectively, the mode function and the natural frequency of transverse vibration of the pre-pressure beam. Based on the boundary conditions with linear elastic support, the following equations are obtained

\[
\phi''(0) = 0, \quad \phi''(L) = 0,
\]

\[K_L \phi(0) + E \phi''(0) + P_0 \phi'(0) = 0, \quad K_R \phi(L) - E \phi''(L) - P_0 \phi'(L) = 0. \]
transverse vibration are illustrated in Fig. 2. Fig. 2 demonstrates that the time histories.

The following equations can be obtained by combining Eqs. (10) and (11)

\[
\beta_1^2 C_1 - \beta_2^2 C_2 = 0,
K_1 C_1 - (E\beta_1^2 - P_0\beta_1) C_2 + K_1 C_3 + (E\beta_2^2 + P_0\beta_2) C_4 = 0,
C_1 \beta_1^2 \cos L\beta_1 + C_2 \beta_2^2 \sin L\beta_1 - C_3 \beta_2^2 \sinh L\beta_2 - C_4 \beta_2^2 \cosh L\beta_2 = 0,
\left[ K_R \cos L\beta_1 - (E\beta_1^2 - P_0\beta_1) \sin L\beta_1 \right] C_1 + \left[ K_R \sin L\beta_1 + (E\beta_1^2 - P_0\beta_1) \cos L\beta_1 \right] C_2
+ \left[ K_R \cosh L\beta_2 - (E\beta_2^2 + P_0\beta_2) \sinh L\beta_2 \right] C_3 + \left[ K_R \sinh L\beta_2 - (E\beta_2^2 + P_0\beta_2) \cosh L\beta_2 \right] C_4 = 0
\]

Then the constant coefficients \( C_j \) \((j = 1, 2, 3, 4)\), the natural frequencies and the corresponding mode functions can be solved. Table 1 presents the physical and geometrical parameter values of a pre-pressure beam with aluminum alloy material [59,60]. In the following study, the parameter values are assigned as listed in Table 1 if there is no special mention.

For asymmetric support boundary conditions with \( K_L = 50000 \, N/m \) and \( K_R = 5000 \, N/m \) and \( K_H = 5000 \, N/m \), the first four natural frequencies of the corresponding linear elastic support beam are determined as \( \omega_1 = 11.749 \, Hz \) (\( \beta_1 = 1.184, \beta_2 = 1.0884 \)), \( \omega_2 = 101.098 \, Hz \) (\( \beta_1 = 3.347, \beta_2 = 3.314 \)), \( \omega_3 = 247.23 \, Hz \) (\( \beta_1 = 5.218, \beta_2 = 5.197 \)), \( \omega_4 = 576.45 \, Hz \) (\( \beta_1 = 7.959, \beta_2 = 7.945 \)). Moreover, the corresponding mode functions of the transverse vibration can be obtained.

The effects of the horizontal spring linear stiffness and the initial axial pre-pressure on the natural frequencies of transverse vibration are illustrated in Fig. 2. Fig. 2 demonstrates that the first four frequencies of the pre-pressure beam decrease as the horizontal spring linear stiffness or the initial axial pressure increases. Fig. 2(a) illustrates that there is a critical stiffness for the horizontal spring. When the fundamental natural frequency of the beam is zero, at the critical stiffness, the support system reaches zero stiffness. If the horizontal spring stiffness continues to increase, the support system will exhibit a negative stiffness. Moreover, higher order natural frequencies linearly decrease with increasing horizontal spring linear stiffness.

4. Primary resonances and force transmissibility

In this paper, the stable steady-state response of the primary resonance of the pre-pressure beam under the distributed excitation is investigated. The Galerkin truncation method and the fourth-order Runge-Kutta are converged to numerically solve the time history of the vibration of the pre-pressure beam. Then the response amplitude is abstracted based on these time histories.

The transverse vibration solutions of Eq. (5) are assumed as \([61–63]\).

\[
U(X, T) = \sum_{k=1}^{n} q_k(T) \psi_k(X)
\]

where \( n \) is an integer greater than 1. \( \psi_k(X) \) represents the trial functions. In this work, the natural mode functions of the corresponding linear beam (11) with the linear spring support (13) are chosen as the trial and weight functions of the Galerkin method. By applying the Galerkin truncation method, the following ordinary differential equations can be derived

<table>
<thead>
<tr>
<th>Item</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus of the beam</td>
<td>( E )</td>
<td>68.9 GPa</td>
</tr>
<tr>
<td>Density of the beam</td>
<td>( \rho )</td>
<td>2800 kg/m³</td>
</tr>
<tr>
<td>Length of the beam</td>
<td>( L )</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Width of the beam</td>
<td>( b )</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Height of the beam</td>
<td>( h_0 )</td>
<td>0.01 m</td>
</tr>
<tr>
<td>Initial axial pressure</td>
<td>( P_0 )</td>
<td>100 N</td>
</tr>
<tr>
<td>External damping of the beam</td>
<td>( C_\theta )</td>
<td>10 N s/m²</td>
</tr>
<tr>
<td>Viscosity damping of the spring</td>
<td>( C_s )</td>
<td>1 N s/m</td>
</tr>
<tr>
<td>Initial length of the horizontal spring</td>
<td>( L_0 )</td>
<td>0.1 m</td>
</tr>
<tr>
<td>Horizontal length of the horizontal spring</td>
<td>( L_H )</td>
<td>0.07 m</td>
</tr>
<tr>
<td>Horizontal spring linear stiffness</td>
<td>( K_H )</td>
<td>5000 N/m</td>
</tr>
<tr>
<td>Vertical spring stiffness at left end</td>
<td>( K_c )</td>
<td>50000 N/m</td>
</tr>
<tr>
<td>Vertical spring stiffness at right end</td>
<td>( K_g )</td>
<td>50000 N/m</td>
</tr>
</tbody>
</table>
\[
\frac{\rho A}{2L} \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k''(X) \right) \frac{dX}{\varphi_m(X)} dX + C_B \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k(X) \varphi_m(X) \right) dX + P_0 \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k''(X) \varphi_m(X) \right) dX
\]
\[
+ \frac{EI}{2L} \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k''(X) \varphi_m(X) \right) dX + \int_0^L \left( F_0 \cos(\Omega T) \varphi_m(X) \right) dX
\]
\[
= \frac{EA}{2L} \int_0^L \left\{ \sum_{k=1}^n q_k(T) \psi_k''(X) \right\} \frac{dX}{\varphi_m(X)} + \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k''(X) \right) \delta(X) \varphi_m(X) dX
\]
\[
- \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k(X) \right) + K_N \left\{ \sum_{k=1}^n q_k(T) \psi_k''(X) \right\} \delta(X) \varphi_m(X) dX
\]
\[
- \int_0^L \left( \sum_{k=1}^n q_k(T) \psi_k(X) \right) + K_N \left\{ \sum_{k=1}^n q_k(T) \psi_k''(X) \right\} \delta(X - L) \varphi_m(X) dX
\]

\( \varphi_m(X) \) denotes the weight functions, where \( m = 1, 2, \ldots, n \). Eq. (17) can be rewritten as
In this paper, the integer \( n \) is set as 4. Eq. (18) can be numerically solved by using the Runge-Kutta method. The initial values of Eq. (18) for the first calculation are set as

\[
q_1 = 0.001, \quad \dot{q}_1 = 0, \quad q_m = 0, \quad \dot{q}_m = 0 \quad \text{for} \quad m = 2, 3, 4
\]  

The response period of the forced vibration can be determined by \( T_d = \frac{2\pi}{\Omega} \). In order to ensure that transient responses have died away, the time history in the time interval of \([0, 200T_d]\) is numerically solved. Moreover, the time interval of \([0, 199T_d]\) is discarded. In order to abstract the steady-state response amplitude of the pre-pressure beam, the local maximum values in the time interval of \([199T_d, 200T_d]\) is recorded.

With \( K_L = 50000 \text{ N/m} \) and \( K_R = 5000 \text{ N/m} \), the steady-state response amplitude and the force transmissibility by the boundaries are described in Figs. 3 and 4. The vertical coordinates \( A_s \) of Figs. 3(a) and 4(a) denotes the steady-state response amplitude. Fig. 3 shows that the 4-term Galerkin truncation method is convergent for the primary resonance of the first three modes. As shown in Fig. 4(b), the force transmission rate of the weaker supporting end may only be larger in the first order resonance region. In the high frequency area, the force transmission rate of the stronger supporting end is significantly higher.

5. Verification by utilizing the finite difference method

In this work, the finite difference method (FDM) is developed to verify the numerical results of the Galerkin truncation method. The nonlinear governing equation and the nonlinear boundary conditions of transverse vibration of the pre-pressure elastic beam with a quasi-zero stiffness support are rewritten as

\[
\rho A \frac{\partial^2 U}{\partial T^2} + EI \frac{\partial^4 U}{\partial X^4} + C_0 \frac{\partial U}{\partial T} + P_0 \frac{\partial^2 U}{\partial X^2} + F_0 \cos(\Omega T) - \frac{EA}{2L} \left( \frac{\partial U}{\partial X} \right)^2 \int_0^L dX = 0
\]  

(a) The first three order resonance

(b) The transmissibility at the left boundary

Fig. 3. Convergence of the truncation order of the Galerkin method.
\[ X = 0 : \frac{\partial^2 U}{\partial x^2} = 0, K_L U + C_S \frac{\partial U}{\partial t} + K_N U^3 + E I \frac{\partial^3 U}{\partial x^3} + P_0 \frac{\partial U}{\partial x} = 0. \]
\[ X = L : \frac{\partial^2 U}{\partial x^2} = 0, K_R U + C_S \frac{\partial U}{\partial t} + K_N U^3 - E I \frac{\partial^3 U}{\partial x^3} - P_0 \frac{\partial U}{\partial x} = 0. \]

(21)

For utilizing the finite difference method, difference formulas as follows are utilized to discrete the continuous space of the pre-pressure beam.
\[
\frac{dU(x,mr)}{dx}
\bigg|_{x=gh} = \frac{1}{12h} \left( U_{g-2}^{m} - 8U_{g-1}^{m} + 8U_{g+1}^{m} - U_{g+2}^{m} \right),
\]
\[
\frac{d^2 U(x,mr)}{dx^2}
\bigg|_{x=gh} = \frac{1}{12h^2} \left( - U_{g-2}^{m} + 16U_{g-1}^{m} - 30U_{g}^{m} + 16U_{g+1}^{m} - U_{g+2}^{m} \right),
\]
\[
\frac{d^3 U(x,mr)}{dx^3}
\bigg|_{x=gh} = \frac{1}{2h^3} \left( - U_{g-2}^{m} + 2U_{g-1}^{m} - 2U_{g+1}^{m} + U_{g+2}^{m} \right),
\]
\[
\frac{d^4 U(x,mr)}{dx^4}
\bigg|_{x=gh} = \frac{1}{h^4} \left( U_{g-2}^{m} - 4U_{g-1}^{m} + 6U_{g}^{m} - 4U_{g+1}^{m} + U_{g+2}^{m} \right)
\]

where \( h \) is the space step. \( \tau \) denotes the temporal step. \( U_g^{m} \) denotes vibration displacement \( U(gh, mr) \), \( m \) presents a positive integer greater than one. \( g = 1, 2, ..., G \), where \( G \) denotes the total step size of discrete space. In this work, \( h \) is set as 0.01. Then Eq. (17) is numerically solved by utilizing the following form

\[
\rho A \frac{U_{g}^{m+1}}{\tau^2} - \frac{2U_{g}^{m} + U_{g}^{m-1}}{h^2} + \left( \frac{EA I_m}{24h^2} \right) \left( - U_{g-2}^{m} + 16U_{g-1}^{m} - 30U_{g}^{m} + 16U_{g+1}^{m} - U_{g+2}^{m} \right) - F_0 \cos(QT) = 0.
\]

When \( m > 1 \), by utilizing the nonlinear support boundary conditions (18), \( U_g^{m} \) can be numerically solved by the following form

\[
U_g^{m+1} = \frac{4\rho A}{2\rho A + C_B \tau} U_g^{m} - \frac{2\tau^2 El}{(2\rho A + C_B \tau)h^2} \left( U_{g-2}^{m} - 4U_{g-1}^{m} + 6U_{g}^{m} - 4U_{g+1}^{m} + U_{g+2}^{m} \right)
\]
\[
+ \frac{2\tau^2 F_0}{2\rho A + C_B \tau} \cos(Qm\tau) = \frac{2\rho A - \tau C_B U_g^{m-1}}{2\rho A + C_B \tau} U_g^{m-1}
\]
\[
+ \frac{2\tau^2 El}{24(2\rho A + C_B \tau)h^2} \left( - U_{g-2}^{m} + 16U_{g-1}^{m} - 30U_{g}^{m} + 16U_{g+1}^{m} - U_{g+2}^{m} \right)
\]

When \( g = 1, 2, G-1, G \), the following equations are adopted

\[
U_g^{m+1} = -U_g^{m} + 2U_{g+1}^{m}, U_g^{m+1} = -U_{g+1}^{m} + 2U_g^{m},
\]
\[
\left( \frac{El}{2h^2} - \frac{P_0}{12h} \right) U_g^{m+2} = \frac{K_L}{U_g^{m}} + C_S \frac{U_g^{m} - U_g^{m-1}}{\tau} + K_N(U_g^{m})^2
\]
\[
+ \left( \frac{El}{h^2} - \frac{2P_0}{3h} \right) U_g^{m+1} + \left( \frac{P_0}{12h} - \frac{El}{h^2} \right) U_g^{m} + \left( \frac{El}{h^2} - \frac{2P_0}{12h} \right) U_g^{m-1}
\]
\[
\left( \frac{El}{2h^2} - \frac{P_0}{12h} \right) U_g^{m+2} = K_R U_g^{m} + C_s \frac{U_g^{m} - U_g^{m-1}}{\tau} + K_N(U_g^{m})^3
\]
\[
+ \left( \frac{El}{h^2} - \frac{2P_0}{3h} \right) U_g^{m+1} + \left( \frac{El}{h^2} - \frac{2P_0}{3h} \right) U_g^{m} + \left( \frac{3h^2}{3h} \right) U_g^{m+1}
\]

The initial conditions for the finite difference calculations can be set as

\[
U_g^0 = 0.001, \quad U_g^0 = 0, \quad U_g^0 = 0.001, \quad U_g^0 = 0 \quad \text{for} \quad g = 1, 2, ..., G
\]

In this work, the temporal step of the finite difference calculations is chosen as \( \tau = 0.0001 \). By utilizing the FDM and the GTM, Fig. 5 presents the comparisons of the steady-state response amplitude and force transmissibility. Fig. 5 shows that near-identical numerical results are obtained. Therefore, the two numerical schemes proposed in this work are both accurate and credible. In particular, the proposed finite difference scheme for the pre-pressure beam with nonlinear support ends can be trusted.
6. Parametric studies

The effects of the geometric nonlinearity on the steady-state response amplitude and the force transmissibility are, respectively presented in Fig. 6(a) and (b). Fig. 6 shows that geometric nonlinearity significantly reduces the response amplitude at resonance. However, the frequency of resonance is also shifted to the high frequency region. Therefore, geometric nonlinearity is beneficial for reducing the resonance intensity but not for the vibration isolation.

Figs. 7 and 8 present the effects of stiffness of the horizontal spring on the resonance intensity and vibration isolation. Under symmetric support, although the influence of the horizontal stiffness on the intensity of the first two primary resonance is opposite, the trend of influence on vibration isolation is the same. As the horizontal stiffness increases, the resonance region shifts to the low frequency region, and the vibration transmission becomes smaller. Therefore, the isolation system, shown in Fig. 1, provides a superior vibration isolation effect. As shown in Fig. 7(c), effective vibration isolation is achieved over a wide frequency range. However, the side effect of high-efficiency isolation is that the amplitude of the first-order primary resonance is too large.

Under asymmetric support conditions, as shown in Fig. 8, the effects of the horizontal spring stiffness on the resonance intensity and the vibration transmissibility of the two ends of the beam are all different. Moreover, the isolation effect of the quasi-zero stiffness system on the elastic vibration of the main structure is affected by modes of the transverse vibration. For a stronger support end, the increase in the stiffness of the horizontal spring will increase the force transmission at the second-order primary resonance. This implies that the quasi-zero stiffness system may increase the vibration transmissibility in certain cases.

Influences of the axial pre-pressure of the beam on the response amplitude and the force transmissibility are presented in Fig. 9. Numerical results show that it is only significant in the first-order modal primary resonance. With the growing of the axial pre-pressure, the amplitude of one end of the asymmetrically supported beam becomes larger, while the amplitude of the other end becomes smaller. On the other hand, the change in the force transmission and the change in amplitude have the same tendency. However, the resonance region moves to the low frequency region, which is beneficial to vibration isolation.

![Fig. 6. Effects of the geometric nonlinearity on steady-state amplitude and force transmissibility.](image)

(a) The steady-state amplitude at the two ends  (b) The transmissibility at the two ends

![Fig. 7. Effects of the horizontal spring linear stiffness with $K_L = K_R = 50000$ N/m.](image)

(a) The first-order primary resonance  (b) The second-order primary resonance  (c) The force transmissibility
7. Conclusions

Nonlinear isolation is firstly investigated for the transverse vibration of continuous structures. The nonlinear isolation system is combined by three linear springs, called as quasi-zero stiffness system. Nonlinear intergro-partial differential governing equation of the elastic beam is derived by considering the geometric nonlinearity. Based on the modal analysis of the pre-pressure beam, the Galerkin truncation is utilized to discretize the governing equation. The stable steady-state response and its transmissibility of the pre-pressure beam under a distributed excitation are numerically solved. Besides, by developing a difference format, numerical results of the Galerkin truncation method are verified by applying the finite difference method. Influences of the axial pre-pressure and nonlinear isolation system on the vibration suppression effect are presented. Then, the following mentionable conclusions can be drawn.

1. The modal analysis finds that there are critical values for the horizontal spring stiffness. At the critical stiffness, the fundamental natural frequency will be zero. With increasing horizontal spring stiffness, higher order natural frequencies decrease linearly.
2. The dynamic response of the nonlinear support elastic beam can be solved by the Galerkin method and the finite difference method.
3. The geometric nonlinearity of the elastic beam is advantageous for the resonance amplitude reduction but not for the vibration isolation.
4. On the whole, this paper demonstrates the superiority of the quasi-zero stiffness vibration isolation system. However, there are certain differences in vibration isolation from discrete systems. When the second-order mode undergoes primary resonance, the quasi-zero stiffness system may increase the force transmission of the vibration of continuous systems. Interestingly, the pre-pressure can be utilized to isolate transverse vibration of continuous systems because it decreases the resonant frequency.

Fig. 8. Effects of the horizontal spring linear stiffness with $K_L = 50000$ N/m; $K_R = 5000$ N/m.
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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jsv.2018.11.028.

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