Irregular instability boundaries of axially accelerating viscoelastic beams with 1:3 internal resonance

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\begin{abstract}
Irregular instability boundaries of axially accelerating beams with 1:3 internal resonance are analytically and numerically investigated in this paper. The distributed parameter is due to the small simple harmonic axial speed. The viscoelastic characteristic of the beam is described by the Kelvin–Voigt model in which the material time derivative is used. A linear partial-differential equation with the variable coefficient and the relevant boundary conditions governing the transverse motion is presented. The effects of the nonhomogeneous boundaries are highlighted. By the method of multiple scales, the solvability conditions in summation and principal parametric resonances are established by some different manipulations in the process of the classical multiple scales method. The Routh–Hurwitz criterion is used to determine the instability boundaries. The effects of viscoelastic coefficient and the viscous damping coefficient are examined on the instability boundaries. Irregular instability boundaries appeared when the 1:3 internal resonance is introduced. It is shown that the numerical calculations by the differential quadrature scheme can verify the approximate analytical results.
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\section{Introduction}
With the rapid development of process industry and precision machinery, the axially moving beams are attracting more and more attention. Although there are advantages of modeling many engineering devices as axially moving beams, there are also many negative factors that limit the use of them, such as, noise and vibration, particularly transverse vibration. Thus, it is important for us to understand the transverse vibration of axially moving beams.

In recent decades, the transverse parametric vibration of axially accelerating beams has been widely researched. Including elastic beams [1–12], and viscoelastic beams [13–22] of linear models. In these studies, the method of multiple scales was widely used in Refs. [2–4, 7, 9, 12–19], the matched asymptotic expansion was applied in Ref. [4], the discretization multi-scale analysis was adopted in Ref. [5], the discretization-averaging analyses was employed in Ref. [11], the direct numerical calculations based on finite difference was used to contrast the approximate analytical solution in Ref. [19], the asymptotic analysis and differential quadrature were introduced in Ref. [20]. At the same time, many researchers have devoted themselves to studying nonlinear parametric vibration of axially accelerating beams [23–35]. The method of multiple scales was used in researching axially accelerating beams [23–25, 29–35], including elastic Euler beams [23, 24], elastic Timoshenko beams [25], viscoelastic Euler beams [29, 30], viscoelastic Rayleigh beams [31], and viscoelastic Timoshenko beams [32–34]. The differential quadrature procedure [35] was used to confirm the approximate analytical solution.

In all of the above-mentioned researches, it was assumed that the beam tensions were uniform along the longitudinal direction [1–35]. If the tensions are equal in size and opposite in directions at both ends of the beam, the acceleration of the beam is zero. As is known of Newton’s second law, a nonzero resultant force leads to the acceleration. Thus, the assumption cannot be entirely correct. In order to address the lack of research in above mentioned aspect, the longitudinal tension variation is first introduced by Chen and Tang to study the combination and principal parametric resonances [36] of Euler beams. Then, they discussed the parametric stability [37] of Euler beams. In addition, Tang et al. investigated Euler beam [38] and Timoshenko beam [39] with the longitudinal variation tension.

In previous works, the multi-scales analysis was used to study axially moving viscoelastic materials with homogeneous boundary conditions. But, the nonhomogeneous boundary conditions are more common in...
engineering. However, there are only a few researchers [38, 40] who investigated the effect of nonhomogeneous boundary conditions.

The internal resonance [41] may occur when two or more natural frequencies are commensurate, and may give rise to the energy exchange between different modes for nonlinear systems. There were lots of researchers [42–50] who studied nonlinear transverse vibration of an axially moving structure with internal resonance. Panda and Kar [51] investigated the nonlinear planar vibration of a pipe conveying pulsatile fluid subjected to principal parametric resonance in the presence of internal resonance. Then, they studied nonlinear dynamics of a hinged-hinged pipe conveying pulsatile fluid subjected to combination and principal parametric resonance in the presence of internal resonance [52]. However, in this paper, it will be found that the internal resonance may lead to strange instability boundaries even the nonlinear terms are ignored.

As a continuation of previously mentioned works [38], four main improvements are presented. In the modeling, the nonlinear term was ignored. In the analysis with the method of multiple scales, the effect of viscoelastic coefficients on the instability boundaries for the first principal parametric resonance are researched in detail. The summation resonance and the second principal parametric resonance are taken into consideration in this paper. The effects of nonhomogeneous boundaries and homogeneous boundaries on the instability boundaries are contrasted.

The rest of this paper is organized as follows. In Section 2, the governing equations and the corresponding nonhomogeneous boundary conditions are given, and the method of multiple scales is adopted. In Section 3, the stability conditions in the summation and principal parametric resonances are explored. In Section 4, the differential quadrature schemes are introduced to verify the analytical results. Section 5 ends the paper.

2. Formulations

In this paper, irregular instability boundaries of an axially accelerating beam with 1:3 internal resonance are investigated. The uniform horizontal beam has density $\rho$, area moment of inertia $I$, Young’s modulus $E$, and cross-sectional area $A$. It travels at the axial transport time-dependent speed $\Gamma(t)$ between two eyelets separated by distance $L$. $t$ is the time. Ignore the geometric nonlinearity of the axially accelerating viscoelastic beam, the governing equation and the associated simply supported boundary conditions are [38]

$$
\rho A \left( v_{tt} + 2 \Gamma v_{tx} + \Gamma^2 v_{xx} \right) - \left[ P_0 + \eta P \Gamma^2 + (x - L) P \Gamma \right] v_{xx} + \int \left[ E v_{xxxx} + a \left( v_{xxxx} + \Gamma v_{xxx} \right) \right] \, dx + k_x v_e + c_4 \left( v_t + \Gamma v_x \right) = 0
$$

(1)

$$
v_{|x=0} = 0, \quad v_{|x=L} = 0; \quad \left[ E v_{xxxx} + a \left( v_{xxxx} + \Gamma v_{xxx} \right) \right]_{|x=0} = 0,
$$

$$
\left[ E v_{xxxx} + a \left( v_{xxxx} + \Gamma v_{xxx} \right) \right]_{|x=L} = 0.
$$

(2)

where $a$ is viscoelastic coefficient, $k_x$ is stiffness of the foundation per unit length, $c_4$ is viscous damping, $\eta$ is axial support stiffness parameter, $P_0$ is initial axial tension (in the beam without the axial acceleration and the transverse vibration). In addition, $v(x, t)$ is the transverse deflection of the beam at position $x$ and time $t$.

To render the governing equation and boundary conditions dimensionless, some special variables and parameters are introduced

$$
v = \frac{\chi}{\sqrt{\chi^2 L}}, \quad x = \frac{x}{L}, \quad t = \frac{t}{\sqrt{\frac{P_0}{\rho A}}}, \quad \Gamma = \frac{\Gamma}{\sqrt{\frac{P_0}{\rho A}}}, \quad E = \frac{E}{\sqrt{\frac{P_0}{\rho A}}}, \quad k_x = \frac{k_x L^2}{P_0}, \quad c_4 = \frac{c_4 L}{\sqrt{\frac{P_0}{\rho A}}}, \quad \kappa = 1 - \eta.
$$

(3)

where bookkeeping parameter $\epsilon$ is a dimensionless positive parameter, it is used to accounting for the fact that the viscoelastic coefficient and the viscous damping coefficient are small. Then, substituting Eq. (3) into Eq. (1) and Eq. (2) leads to

$$
v_{tt} + 2 \Gamma v_{tx} + \left[ \kappa \chi^2 - (\chi - 1)^2 \right] v_{xx} + k_x v = -\epsilon c_4 \left( v_t + \gamma v_x \right) - \epsilon a \left( v_{xxxx} + \gamma v_{xxx} \right)
$$

(4)

$$
\left. v_{|x=0} = 0; \quad v_{|x=1} = 0; \quad \left[ \kappa \chi^2 v_{xx} + \epsilon a \left( v_{xx} + \gamma v_{xx} \right) \right]_{|x=0} = 0,
$$

$$
\left. \left[ \kappa \chi^2 v_{xx} + \epsilon a \left( v_{xx} + \gamma v_{xx} \right) \right]_{|x=1} = 0.
$$

(5)

The axial transport speed is assumed to be a small simple harmonic variation about the constant mean axial speed $\gamma_0$ and an expansion of displacement, namely

$$
\gamma(t) = \gamma_0 + \epsilon \gamma_1 \sin(\omega t)
$$

(6)

where $\epsilon \gamma_1$ and $\omega$ denote the amplitude and frequency of fluctuation respectively. Substituting Eq. (6) into Eqs. (4) and (5) yield

$$
v_{tt} + 2 \Gamma_0 v_{tx} + \left( \kappa \chi^2_0 - 1 \right) v_{xx} + k_x v = -\epsilon c_4 \left( v_t + \gamma_0 v_x \right) + a \left( v_{xxxx} + \gamma_0 v_{xxx} \right)
$$

$$
+2 \gamma_1 \sin(\omega t) \left( v_{tx} + \kappa \chi_0^2 v_{xx} \right)
$$

$$
+ \left( 1 - x \right) \eta \gamma_1 \cos(\omega t) v_{xx} + O(\epsilon^2)
$$

(7)

$$
\left. v_{|x=0} = 0; \quad v_{|x=1} = 0; \quad \left[ \kappa \chi_0^2 v_{xx} + \epsilon a \left( v_{xx} + \gamma_0 v_{xx} \right) + O(\epsilon^2) \right]_{|x=0} = 0,
$$

$$
\left. \left[ \kappa \chi_0^2 v_{xx} + \epsilon a \left( v_{xx} + \gamma_0 v_{xx} \right) + O(\epsilon^2) \right]_{|x=1} = 0.
$$

(8)

In mathematics and physics, the method of multiple scales comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent. The solutions to Eq. (7) can be assumed as

$$
v(x, t; \epsilon) = v_0(x, \tau_0, T_1) + v_1(x, \tau_0, T_1) + O(\epsilon^2)
$$

(9)

where $\tau_0 = t$ and $T_1 = \epsilon t$ respectively indicate the fast and slow time scales. Substituting Eq. (9) and the following relationship

$$
\frac{\partial}{\partial \tau_0} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial T_1}, \quad \frac{\partial^2}{\partial \tau_0^2} = \frac{\partial^2}{\partial t_0^2} + 2 \epsilon \frac{\partial}{\partial T_0} + O(\epsilon^2)
$$

(10)

into (7) and (8), collecting the coefficient terms of $\epsilon^0$ and $\epsilon^1$ on both sides in the resulting equations, they produce the zero order operator of

$$
v_{0, \tau_0 \tau_0} + 2 \gamma_0 v_{0, \tau_0 \tau_0} + \left( \kappa \chi^2_0 - 1 \right) v_{0, \tau_0 \tau_0} + k_x v_0 = 0
$$

(11)

$$
v_{0, |x=0} = 0; \quad v_{0, |x=1} = 0; \quad \left. k_x^2 v_{0, \tau_0 \tau_0} \right|_{x=0} = 0, \quad \left. k_x^2 v_{0, \tau_0 \tau_0} \right|_{x=1} = 0.
$$

(12)

the first order operator of

$$
v_{1, \tau_0 \tau_0} + 2 \gamma_0 v_{1, \tau_0 \tau_0} + \left( \kappa \chi^2_0 - 1 \right) v_{1, \tau_0 \tau_0} + k_x v_1 = -c_4 \left( v_{0, \tau_0 \tau_0} + \gamma_0 v_{0, \tau_0 \tau_0} \right) - a \left( v_{0, \tau_0 \tau_0 \tau_0} + \gamma_0 v_{0, \tau_0 \tau_0 \tau_0} \right)
$$

$$
-2 \left( v_{0, \tau_0 \tau_0} + \gamma_0 v_{0, \tau_0 \tau_0} \right) \left( -1 - x \right) \eta \gamma_1 \cos(\omega t) v_{xx}
$$

$$
-2 \gamma_1 \sin(\omega t) \left( v_{xx} + \kappa \chi_0^2 v_{xx} \right)
$$

(13)

$$
\left. v_{1, |x=0} = 0; \quad v_{1, |x=1} = 0; \quad \left. k_x^2 v_{1, \tau_0 \tau_0} + a \left( v_{0, \tau_0 \tau_0} + \gamma_0 v_{0, \tau_0 \tau_0} \right) \right|_{x=0} = 0,
$$

$$
\left. \left[ k_x^2 v_{1, \tau_0 \tau_0} + a \left( v_{0, \tau_0 \tau_0} + \gamma_0 v_{0, \tau_0 \tau_0} \right) \right]_{|x=1} = 0.
$$

(14)
The solution of Eq. (12) can be written as

$$v_0(x, T_0, T_1) = \sum_{n=0}^{\infty} A_n(T_1) \varphi_n(x)e^{i\omega_T T_0} + cc$$  \hspace{1cm} (15)

where $A_n$ is a complex function of undetermined. $cc$ denotes complex conjugate of the preceding terms.

Under the Eq. (15) and the boundary conditions (12), the nth complex frequency $\lambda_n = \delta_n + i\omega_n$ can be solved

$$= \left[ (\rho_n - P_n^1) (\rho_n - P_n^2) (\rho_n - P_n^3) (\rho_n - P_n^4) \right] \left[ (\rho_n - P_n^1) (\rho_n - P_n^2) (\rho_n - P_n^3) (\rho_n - P_n^4) \right] = 0$$  \hspace{1cm} (16)

The nth complex mode function $\varphi_n$ is

$$\varphi_n(x) = C_n \left\{ \frac{\rho_n - P_n^1}{P_n - P_n^1} e^{\rho_n x} + \frac{\rho_n - P_n^2}{P_n - P_n^2} e^{\rho_n x} - \frac{\rho_n - P_n^3}{P_n - P_n^3} e^{\rho_n x} + \frac{\rho_n - P_n^4}{P_n - P_n^4} e^{\rho_n x} \right\}$$  \hspace{1cm} (17)

If the dimensionless parameters are given as $k_1 = 0.2$, $k_2 = 0.72$, $\kappa = 0.5$, and $\gamma_0 = 0.68874$, it can be calculated that the first two natural frequencies of the linear generating system are $\omega_1 = 3.225355493$ and $\omega_2 = 9.676049245$. It can be easily found that $\omega_2$ approaches three times of $\omega_1$. Thus, a three-to-one internal resonance may occur.

3. Stability

3.1. Stability of summation parametric resonance

In this section, the detuning parameters $\sigma_1$ and $\sigma_2$ are introduced to quantify the nearness of $\omega_2$ to $3\omega_1$ and $\omega$ to $(\omega_1 + \omega_2)$. Thus, the frequency relations for the 1:3 internal resonance and the summation parametric resonance can be written as

$$\omega_2 = 3\omega_1 + \epsilon \sigma_1, \hspace{1cm} \omega = \omega_1 + \omega_2 + \epsilon \sigma_2$$  \hspace{1cm} (18)

The solution of Eq. (13) can be assumed as

$$v_0(x, T_0, T_1) = \varphi_1(x)A_1(T_1)e^{i\omega_T T_0} + \varphi_2(x)A_2(T_1)e^{i\omega_T T_0} + cc$$  \hspace{1cm} (19)

where $A_1(T_1)$ and $A_2(T_1)$ are undetermined functions, $cc$ is a complex symbol which denotes the complex conjugate of the two preceding terms on the right hand of the equation. Substituting Eq. (19) into Eq. (13) yields

$$v_1 = \omega_1 v_0 + \gamma_1 \varphi_1 + \gamma_2 \varphi_2 \hspace{1cm} \text{at} \hspace{1cm} T_1 = \omega_1 T_0 + \omega_2 T_0 + \omega_1 T_0$$

where the dot denotes the derivatives of $T_1$, NST is non-secular terms or small divisor terms, and

$$\xi_0 = \omega_1 \varphi_1 + \gamma_1 \varphi_1, \hspace{1cm} \xi_1 = \omega_1 \varphi_1 + \gamma_1 \varphi_1, \hspace{1cm} \xi_2 = \omega_1 \varphi_1 + \gamma_1 \varphi_1$$

To investigate the solvability condition with the nonhomogeneous boundary conditions, the solution of Eq. (13) can be assumed as

$$v_1(x, T_0, T_1) = \varphi_1(x, T_1)e^{i\omega_T T_0} + \varphi_2(x, T_1)e^{i\omega_T T_0} + N(x, T_0, T_1) + cc$$  \hspace{1cm} (22)

where $N(x, T_0, T_1)$ is all non-secular terms in the solution. Substituting Eq. (22) into Eq. (20), collecting coefficients of $\exp(i\omega_1 T_0)$ and $\exp(i2\omega_2 T_0)$ in the resulting equation lead to

$$= \left[ 2\varphi_1 A_1 + (\epsilon \varphi_1 + \omega_1 A_1) + \gamma_1 \varphi_2 A_2 e^{i\omega_T T_0} \right]$$

$$\gamma_1 \varphi_1 + \gamma_2 \varphi_2$$

where $\omega_1 = 0.2$, $\omega_2 = 0.72$, $\kappa = 0.5$, and $\gamma_0 = 0.68874$, it can be calculated that the first two natural frequencies of the linear generating system are $\omega_1 = 3.225355493$ and $\omega_2 = 9.676049245$. It can be easily found that $\omega_2$ approaches three times of $\omega_1$. Thus, a three-to-one internal resonance may occur.

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where $A_1(T_1)$ and $A_2(T_1)$ are undetermined functions, $cc$ is a complex symbol which denotes the complex conjugate of the two preceding terms on the right hand of the equation. Substituting Eq. (19) into Eq. (13) yields

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where the dot denotes the derivatives of $T_1$, NST is non-secular terms or small divisor terms, and

$$\xi_0 = \omega_1 \varphi_1 + \gamma_1 \varphi_1, \hspace{1cm} \xi_1 = \omega_1 \varphi_1 + \gamma_1 \varphi_1, \hspace{1cm} \xi_2 = \omega_1 \varphi_1 + \gamma_1 \varphi_1$$

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where $N(x, T_0, T_1)$ is all non-secular terms in the solution. Substituting Eq. (22) into Eq. (20), collecting coefficients of $\exp(i\omega_1 T_0)$ and $\exp(i2\omega_2 T_0)$ in the resulting equation lead to

$$= \left[ 2\varphi_1 A_1 + (\epsilon \varphi_1 + \omega_1 A_1) + \gamma_1 \varphi_2 A_2 e^{i\omega_T T_0} \right]$$

$$\gamma_1 \varphi_1 + \gamma_2 \varphi_2$$

where $\omega_1 = 0.2$, $\omega_2 = 0.72$, $\kappa = 0.5$, and $\gamma_0 = 0.68874$, it can be calculated that the first two natural frequencies of the linear generating system are $\omega_1 = 3.225355493$ and $\omega_2 = 9.676049245$. It can be easily found that $\omega_2$ approaches three times of $\omega_1$. Thus, a three-to-one internal resonance may occur.
\[ \begin{align*}
\dot{p}_2 &= -\gamma_1^2 p_1 - \gamma_1^2 q_1 - (0.5c_d + a_2^2) p_2 + S_1 q_2 \\
\dot{q}_2 &= -\gamma_1^2 p_1 + \gamma_1^2 q_1 - S_2 p_2 - (0.5c_a + a_1^2) q_2 \\
\end{align*} \]
\[ (31) \]

The characteristic equation of the Jacobian matrix of Eq. (31) is
\[ [J] = 
\begin{bmatrix}
-0.5c_d + a_1^2 & S_1 & -\gamma_1^2 & -\gamma_1^2 \\
-S_1 & -0.5c_d + a_1^2 & -\gamma_1^2 & -\gamma_1^2 \\
-\gamma_1^2 & -\gamma_1^2 & -0.5c_a + a_2^2 & S_2 \\
S_2 & -\gamma_1^2 & -\gamma_1^2 & -0.5c_a + a_2^2 \\
\end{bmatrix} \]
\[ (32) \]

The eigenvalues can be obtained as
\[ \lambda^4 + b_1 \lambda^3 + b_2 \lambda^2 + b_3 \lambda + b_4 = 0 \]
\[ (33) \]
where
\[ D_1 = \frac{c_d}{2a} + \xi_1, \quad D_2 = \frac{c_a}{2a} + \xi_1, \]
\[ b_1 = 2a(D_1 + D_2), \]
\[ b_2 = S_1^2 + S_2^2 + a^2(D_1^2 + 4D_1 D_2 + D_2^2) - 2\gamma_1^2 S_1 \xi_2 \xi_1, \]
\[ b_3 = 2\left[ 2a(D_1^2 S_1^2 + D_2^2 S_2^2) + a^2 D_1 D_2 (D_1 + D_2) \right. \]
\[ -\xi_1^2 (D_1 + D_2) S_1 (S_1 - S_2) \xi_2 \xi_1 + S_1 \xi_2 \xi_1 \xi_2 \right] \]
\[ b_4 = \left( S_1^2 + a^2 D_1^2 \right) (S_2^2 + a^2 D_2^2) + a^2 (S_1 S_2)^2 \xi_2 \xi_1 + 2a D_1 D_2 S_1 S_2 \xi_2 \xi_1 \xi_2 \xi_1 \]
\[ -2a^2 D_1 D_2 S_1 S_2 R(\xi_1^2) + (D_1 D_2) R(\xi_2^2) + 2S_1 S_2 \xi_2 R(\xi_2^2) + \left| \xi_2 \right|^2 \xi_2^2 R_1 \xi_1 \xi_1 \quad (34) \]

In control system theory, the Routh–Hurwitz criterion is a necessary and sufficient condition for the stability. The Routh–Hurwitz criterion gives the trivial solution to Eq. (33) as
\[ \Delta_1 = b_1 > 0, \quad \Delta_2 = b_1^2 \begin{bmatrix} 1 & 0 \\ b_2 & b_3 \end{bmatrix} > 0, \quad \Delta_3 = b_3 \begin{bmatrix} 1 & 0 \\ b_1 & b_2 \end{bmatrix} > 0, \quad \Delta_4 = b_4 > 0 \quad (35) \]

The stability condition of inequalities (35) is \( |\gamma_1| < \gamma_{\text{min}} \).

In order to study 3:1 internal resonance, some fixed parameters are selected in the numerical examples as follows: \( k_2 = 0.2, k_3 = 0.72, k = 0.5, a = 0.00001, c_d = 0.0001, \) and \( \gamma_0 = 0.86, \) if no other values are noted. Then, the corresponding two natural frequencies are \( \omega_0 = 2.9171 \) and \( \omega_{02} = 9.4122, \) and the internal frequency detuning parameter is \( \sigma_1 = 0.6609. \)

Fig. 1 shows the effects of the viscoelastic coefficients on the stability boundaries for the summation parametric resonances in plane \( \sigma_2 - \gamma_1. \)

The solid line denotes \( \alpha = 0, \) the dashed line denotes \( \alpha = 0.00001, \) and the dash-dot line denotes \( \alpha = 0.0001. \) It can be found that the stability boundaries are bilaterally symmetric about \( \sigma_2 \) axis. The effect of the viscoelastic coefficients on the stability boundary are complex. For the small \( \sigma_2, \) the larger viscoelastic coefficient leads to the larger instability threshold of \( \gamma_1 \) for given \( \sigma_2, \) and the smaller instability range of \( \sigma_2 \) for given \( \gamma_1. \) However, the trend is reversed for large \( \sigma_2. \) Because of the viscoelastic coefficient, growth of the lines has gradually become slow.

Fig. 2 depicts the effects of the viscous damping coefficients on the stability boundaries for the summation parametric resonances in plane \( \sigma_2 - \gamma_1. \) The solid line denotes \( \zeta_d = 0.001, \) the dashed line denotes \( \zeta_d = 0.1, \) and the dash-dot line denotes \( \zeta_d = 0.5. \) It is revealed that the larger viscous damping coefficient has the effect of narrowing the instability zones of \( \sigma_2 \) for given \( \gamma_1 \) and increasing the instability zones of \( \gamma_1 \) for given \( \sigma_2. \) That is, the increasing viscous damping coefficient makes the instability boundaries move toward the increasing direction of \( \gamma_1 \) in plane \( (\omega, \gamma_1) \) and the instability regions become narrow.

### 3.2 Stability of the first principal parametric resonance

In this section, we study the cases of the 1:3 internal and the first principal parametric resonances of first two modes. The relations can be written as
\[ \omega_2 = 3\omega_1 + \varepsilon \sigma_1, \quad \omega = 2\omega_1 + \varepsilon \sigma_2. \quad (36) \]

The solution to Eq. (13) can be expressed as
\[ \begin{align*}
t_0(t, T_0, T_1) &= \varphi_1(t) A_1(T_1) e^{i\omega_0 T_0} + \varphi_2(t) A_2(T_1) e^{i\omega_2 T_0} + cc \\
t_1(t, T_0, T_1) &= \varphi_1(t, T_1) e^{i\omega_0 T_0} + \varphi_2(t, T_1) e^{i\omega_2 T_0} + N(t, T_0, T_1) + cc
\end{align*} \]
\[ (37) \]

Applying the same steps, we can obtain
\[ \begin{align*}
\sigma_1 + (0.5c_d + a_1^2) A_1 + \gamma_1 \xi_2 \xi_1 A_2 e^{i(\sigma_2 - \sigma_1) T_1} + \gamma_1 \xi_2 \xi_1 A_1 e^{i\omega_2 T_1} &= 0 \quad (38a) \\
\sigma_2 + (0.5c_d + a_1^2) A_2 + \gamma_1 \xi_2 \xi_1 A_1 e^{i(\sigma_2 - \sigma_1) T_1} &= 0 \quad (38b)
\end{align*} \]
where
\[ \begin{align*}
\xi_1 &= \int_{T_0}^{T_0 + T_1} \xi_1 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_2 &= \int_{T_0}^{T_0 + T_1} \xi_2 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_3 &= \int_{T_0}^{T_0 + T_1} \xi_3 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_4 &= \int_{T_0}^{T_0 + T_1} \xi_4 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_5 &= \int_{T_0}^{T_0 + T_1} \xi_5 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_6 &= \int_{T_0}^{T_0 + T_1} \xi_6 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt
\end{align*} \]
\[ (39a) \]
\[ \begin{align*}
\xi_7 &= \int_{T_0}^{T_0 + T_1} \xi_7 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_8 &= \int_{T_0}^{T_0 + T_1} \xi_8 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_9 &= \int_{T_0}^{T_0 + T_1} \xi_9 \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt \\
\xi_{10} &= \int_{T_0}^{T_0 + T_1} \xi_{10} \phi_1(t) dt + \gamma_0 \phi_1(t) \phi_1(t) \phi_2(t) \left| \phi_1(t) \right|^2 dt
\end{align*} \]
\[ (39b) \]

It is numerically confirmed that \( \xi_1 \) and \( \xi_2 \) are positive real numbers; \( \xi_3, \xi_5 \) and \( \xi_6 \) are complex numbers.
In order to cast Eq. (38) into an autonomous system, introduce the transformation

\[ A_1(T_1) = [p_1(T_1) + iq_1(T_1)] e^{\lambda_1 T_1}, \quad A_2(T_1) = [p_2(T_1) + iq_2(T_1)] e^{\lambda_2 T_1}, \]

where \( p_0(T_1) \) and \( q_0(T_1) \) are real functions, and

\[ S_1 = \frac{1}{2} \sigma_2, \quad S_2 = \frac{1}{2} \sigma_2 - \sigma_1. \]

Substituting Eq. (40) into Eqs. (38a) and (38b), simplifying and separating the real and imaginary parts in the resulting equations lead to

\[
\begin{align*}
p_1 &= -((0.5c_2 + \alpha_1 \gamma_1) p_1 + (S_1 - \gamma_1 c_2^2) q_1 - \gamma_1 c_2^2 q_2) \\
q_1 &= -(S_1 + \gamma_1 c_2^2) p_1 - (0.5c_2 + \alpha_1 \gamma_1) q_1 - \gamma_1 c_2^2 p_2 - \gamma_1 c_2^2 q_2 \\
p_2 &= -\gamma_1 c_2^2 p_1 + \gamma_1 c_2^2 q_1 - (0.5c_2 + \alpha_2 \gamma_1) p_2 + S_2 q_2 \\
q_2 &= -\gamma_1 c_2^2 p_1 - \gamma_1 c_2^2 q_1 - S_2 p_2 - (0.5c_2 + \alpha_2 \gamma_1) q_2
\end{align*}
\]

The characteristic equation of the Jacobian matrix is

\[ x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4 = 0 \]

where

\[
\begin{align*}
b_1 &= 2 [c_2 + a (\gamma_1 + \xi_1)] \\
b_2 &= S_1^2 + S_2^2 - \gamma_1^2 |\xi_1|^2 + [c_2 + a (\gamma_1 + \xi_1)]^2 \\
&\quad + 2 (0.5c_2 + a \gamma_1) (0.5c_2 + a \xi_1) - 2 \gamma_1^2 \text{Re}(\xi_2 \xi_3), \\
b_3 &= 2 \left( S_1^2 \xi_1 + S_2^2 \xi_2 + [c_2 + a (\gamma_1 + \xi_1)] (0.5c_2 + a \gamma_1) (0.5c_2 + a \xi_1) \\
&\quad - \gamma_1^2 \text{Re}(\xi_2 \xi_3) - \gamma_1^2 (S_1 + S_2) \text{Im}(\xi_2 \xi_3) - \gamma_1^2 |\xi_1|^2 (0.5c_2 + a \xi_1), \\
b_4 &= (S_1 S_2 - \gamma_1^2 |\xi_1|^2 + [S_1 (0.5c_2 + a \xi_1) - \gamma_1^2 |\xi_1|^2]^2 \\
&\quad + (0.5c_2 + a \gamma_1) (0.5c_2 + a \xi_1) - \gamma_1^2 |\xi_1|^2)^2 \\
&\quad + 2 \gamma_1^2 \left(0.5c_2 + a \xi_1\right) \xi_1^2 - S_1 \xi_2^2 (0.5c_2 + a \xi_1)\xi_2^2 - S_2 \xi_2^2 \right].
\end{align*}
\]

The Routh–Hurwitz criterion of the trivial solution is introduced as

\[ \Delta_1 = b_1 > 0; \quad \Delta_2 = \left|\begin{array}{cc} b_1 & 1 \\ b_2 & b_1 \end{array}\right| > 0, \quad \Delta_3 = \left|\begin{array}{ccc} b_1 & 1 & 0 \\ b_2 & b_1 & b_1 \\ 0 & b_1 & b_1 \end{array}\right| > 0, \quad \Delta_4 = b_4 > 0 \]

The stability condition is \( |\gamma_1| < \gamma_{\text{min}} \).

It can be found that the stability boundaries are complicated when \( a = 0 \) and \( c_2 = 0.1 \) in Tang’s paper [38]. As a further study, we detailed research the effect of the small viscoelastic coefficients on the stability boundaries. Fig. 3 shows the effects of the viscosity coefficients on the stability boundaries for the principal parametric resonance in plane \( \sigma_2-\gamma_1 \). The solid line denotes \( a = 0.000001 \), the dash-dot line denotes \( a = 0.00001 \), and the dotted line denotes \( a = 0.0001 \). The regions above the boundaries are instable and the regions below the presented boundaries are stable. The stability boundary is bilateral symmetry about \( \sigma_2 \) axis when \( a = 0.0001 \), but others are no bilateral symmetry. The change of the stability boundaries are complicated when \( a = 0, a = 0.000001, \) and \( a = 0.00001 \). Fig. 3(b) depicts the variation of the stability boundaries in detail. There are sinking areas of the stability boundaries between \( \sigma_2 = 0.32 \) and \( \sigma_2 = 0.7 \). The response has multiple solution in this area. It is revealed that the larger viscoelastic coefficient have the effect of narrowing the instability zones of \( \sigma_2 \) for given \( \gamma_1 \) and increasing the instability zones of \( \gamma_1 \) for given \( \sigma_2 \). The sinking area disappeared with increased viscosity coefficient.

Fig. 4 illustrates the effects of the viscous damping coefficients on the stability boundaries for the principal parametric resonance in plane \( \sigma_2-\gamma_1 \). The solid line denotes \( c_d = 0.001 \), the dashed line denotes \( c_d = 0.1 \), and the dash-dot line denotes \( c_d = 0.5 \). The larger viscous damping coefficients have the effect of narrowing the instability zones of \( \sigma_2 \) for given \( \gamma_1 \) and increasing the instability zones of \( \gamma_1 \) for given \( \sigma_2 \). Namely, the increasing viscous damping coefficient makes the stability boundaries move toward the increasing direction of \( \gamma_1 \) and the instability regions become narrow. Comparing Figs. 2 and 4, it can be found that the principal parametric resonances and the summation parametric...
resonances have the same tendencies. The stability boundaries in the principal resonances are more sensitive to the viscous damping coefficient than in the summation resonances.

Fig. 5(a) and (b) show the instability boundaries of the parametric resonance with internal resonance and that without internal resonance (related terms are dropped). In Fig. 5(a) and (b), $\alpha = 10^{-6}$ and $\alpha = 10^{-5}$. The solid line denotes the instability boundaries of the parametric resonance with internal resonance, the dashed line denotes that without internal resonance. Comparing the solid line and the dashed line, it’s obvious that instability boundaries are regular without internal resonance and irregular with internal resonance. Therefore, it denotes that the internal resonance leads to strange instability boundaries.

3.3. Stability of the second principal parametric resonance

In this section, we study the cases of the 1:3 internal and the second principal parametric resonances of first two modes. The relations can be expressed as

$$\omega_2 = 3\omega_1 + \varepsilon\sigma_1, \quad \omega = 2\omega_2 + \varepsilon\sigma_2.\quad (46)$$

The solution to Eq. (13) can be expressed by

$$\nu_1(x, T_0, T_1) = \varphi_1(x) A_1(T_1) e^{i\omega T_0} + \varphi_2(x) A_2(T_1) e^{i\omega T_0} + cc$$

$$\nu_1(x, T_0, T_1) = \varphi_1(x, T_1) e^{i\omega T_0} + \varphi_2(x, T_1) e^{i\omega T_0} + N(x, T_0, T_1) + cc\quad (47)$$

Applying the same steps, we can obtain

$$\alpha A_1 \nu_1 |_{\nu_1, x, x} = \left(-[2\zeta_0 A_1 + (\varepsilon_0 \nu_0 + \alpha \zeta_1) A_1] \varphi_1 \right)\quad (48a)$$

$$\alpha A_2 \varphi_2 |_{\nu_2, x, x} = \left(-[2\zeta_0 A_2 + (\varepsilon_0 \nu_0 + \alpha \zeta_1) A_2 + \gamma_1 \zeta_2 \varphi_2 e^{i\omega T_1}] \varphi_2 \right)\quad (48b)$$

where

$$\zeta_0 = i\omega_1 \nu_1 + \gamma_0 \varphi_1', \quad \zeta_1 = i\omega_1 \varphi_1'' + \gamma_0 \varphi_1''',$$

$$\varphi_2 = i\omega_2 \varphi_2 + \gamma_0 \varphi_2'', \quad \varphi_2 = i\omega_2 \nu_2'' + \gamma_0 \nu_2''',$$

$$\zeta_2 = \left(1 - \chi \omega_2 - i\zeta_0\right)\varphi_2'' - \omega_2 \varphi_2'.\quad (49)$$

The complex variable modulation equations for amplitude and phase can be obtained as

$$A_1 + (0.5c_4 + \alpha \xi_1) A_1 = 0\quad (50a)$$

$$A_2 + (0.5c_4 + \alpha \xi_1) A_2 + \gamma_1 \zeta_2 A_2 e^{i\omega T_1} = 0\quad (50b)$$

where

$$\alpha_1 = \int_0^1 \zeta_1 \varphi_1 d x + \gamma_0 \varphi_1 x = A_1|_0^1, \quad \zeta_0 = 2\zeta_0 \varphi_1 d x$$

$$\alpha_2 = \int_0^1 \zeta_1 \varphi_2 d x + \gamma_0 \varphi_2 x = A_2|_0^1, \quad \zeta_0 = 2\zeta_0 \varphi_2 d x$$

It is numerically confirmed that $\zeta_1$ and $\zeta_1$ are positive real numbers, $\zeta_2$ is complex number. Thus, it can be found that the first mode has actually no effect on the dynamic stability in principal parametric resonance of the second mode.

The transformation is introduced as

$$A_2(T_1) = [p_2(T_1) + i q_2(T_1)] e^{i\omega T_1}.\quad (52)$$

where $p_2$ and $q_2 (h=1, 2)$ are real functions, and

$$S_2 = \sigma_2^2.\quad (53)$$

Substituting Eq. (52) into Eq. (50b), simplifying and separating the real and imaginary parts in the resulting equations yield

$$p_2 = -(0.5c_4 + \alpha \xi_1 + \gamma_1 \xi_2^R) p_2 + (S_2 - \gamma_1 \xi_2^I) q_2$$

$$q_2 = -(S_2 - \gamma_1 \xi_2^I) p_2 - (0.5c_4 + \alpha \xi_1 - \gamma_1 \xi_2^R) q_2\quad (54)$$

The characteristic equation of the Jacobian matrix of Eq. (54) is written as

$$[J] = \begin{pmatrix}
-0.5c_4 + \alpha \xi_1 + \gamma_1 \xi_2^R & S_2 - \gamma_1 \xi_2^I \\
-S_2 + \gamma_1 \xi_2^I & -0.5c_4 + \alpha \xi_1 - \gamma_1 \xi_2^R
\end{pmatrix}\quad (55)$$

The eigenvalues can be obtained as

$$\lambda^2 + 2aD_2 \lambda\left(S_2 + a^2 D_2 - \gamma_1 \xi_2^I \right)^2 = 0\quad (56)$$

where

$$D_2 = \frac{\xi_2^I}{2a} + \gamma_1\quad (57)$$

According to the Routh–Hurwitz criterion, the stability condition is

$$|\gamma_1| < \frac{a^2 D_2^2 + S_2^2}{|\xi_2^I|^2}\quad (58)$$

Fig. 6 shows the stability boundaries for the second principal parametric resonance in plane $\sigma_2 - \gamma_1$ for the different viscoelastic coefficients. The solid line denotes $a = 0$, the dashed line denotes $a = 0.00001$, and the dash-dot line denotes $a = 0.0001$. The larger viscoelastic coefficients have the effect of narrowing the instability zones of $\sigma_2$ for given $\gamma_1$ and increasing the instability zones of $\gamma_1$ for given $\sigma_2$. Comparing
The effects of the viscoelastic coefficients on the stability boundaries.

Fig. 6. The effects of the viscous damping coefficients on the stability boundaries.

Fig. 7. The effects of the viscous damping coefficients on the stability boundaries.

Fig. 8. The comparison of the analytical results and the numerical results for the summation parametric resonance.

Fig. 9. The comparison of the analytical results and the numerical results for the first principal parametric resonance.

4. Comparisons of analytical results and numerical integrations

The differential quadrature scheme [53] will be used to solve numerically the governing equation. Consider the domain \( x \in [0, 1] \), \( N \) are the numbers of sampling points in \( x \) direction and \( \varepsilon = 1 \). In this paper, the same method and sampling points are selected with Tang et al. [38].

4. Comparisons of analytical results and numerical integrations

Thus,

\[
\eta_i + \sum_{k=1}^{N-1} \left( 2\gamma A^{(1)}_{ik} + \alpha A^{(2)}_{ik} \right) \psi_k + c_d \psi_i + k_i \psi_i \\
+ \sum_{k=1}^{N-1} \left[ \kappa^2 \left( x_i - (x - 1) \right)^2 \right] \psi_k + k_i \psi_i + c_d \psi_i + a \gamma A^{(2)}_{ik} \right] \psi_k = 0 \\
(\varepsilon = 2, 3, \ldots, N - 1) \tag{59}
\]

For \( N = 21 \), \( k_2 = 0.2 \), \( k_3 = 0.72 \), \( \kappa = 0.5 \), \( \gamma_0 = 0.86 \), \( c_d = 0.001 \), and \( a = 0.00001 \), Fig. 8 presents the stability boundaries for the summation parametric resonance in plane \( \sigma_2 - \gamma \). The solid line denotes the results with nonhomogeneous boundary conditions (RNHBC), the dashed line denotes the results with homogeneous boundary conditions (RHBC), and the dot line denotes the numerical results (NR). Comparing the solid line with the dashed line, it can be found that the results with homogeneous boundary conditions have larger stability zones. It shows that the numerical calculations have a good agreement with the analytical results with homogeneous boundary conditions.

Fig. 9 shows the stability boundaries for the first principal parametric resonance in plane \( \sigma_2 - \gamma \). The solid line denotes the results with...
Fig. 10. The comparison of the analytical results and the numerical results for the second principal parametric resonance.

nonhomogeneous boundary conditions, the dashed line denotes the results with homogeneous boundary conditions, and the dot line denotes the numerical results. As can be seen in Fig. 9, the numerical results, the results with nonhomogeneous boundary conditions, and the results with homogeneous boundary conditions instability boundary have a reasonable agreement in most areas. As well as, there are numerical differences among these three methods in special part, but the same tendencies. The three cases have the same qualitative results. On the other hand, if we want to get more accurate results, much more computationally time should be spent. So we just investigate the comparison of analytical and numerical results through qualitative analysis.

Fig. 10 illustrates the stability boundaries for the second principal parametric resonance in plane $\sigma_2 - \gamma_1$. The solid line denotes the results with nonhomogeneous boundary conditions, the dashed line denotes the results with homogeneous boundary conditions, and the dot line denotes the numerical results. As can be seen from the figure, the numerical results are nearly equal to the results with nonhomogeneous boundary conditions, and the results with homogeneous boundary conditions. The homogeneous boundary conditions and the nonhomogeneous boundary conditions have little different effect for the second principal parametric resonance. The results of both the numerical calculations and the analytical results not only yield qualitatively the same results, but also quantitatively close.

5. Conclusions

The linear vibrations of axially accelerating viscoelastic beams subjected to parametric and 1:3 internal resonance are studied in detail in this paper. The effect of the nonhomogeneous boundary is highlighted. The method of multiple scales is used to establish the solvability conditions. The Routh–Hurwitz criterion is used to determine the stabilities of the steady-state responses. The effects of viscoelastic coefficient and viscous damping coefficient are examined on the stability boundaries. Special instability boundaries appear when 1:3 internal resonance is introduced. The numerical calculations are used to illustrate the approximate analytical results. Some important conclusions are listed as follows:

1. Higher values of the viscous damping coefficient have the effect of raising and narrowing the instability zones for the parametric resonances;
2. The internal resonance leads to strange instability boundaries even the nonlinear terms are ignored. The viscoelastic coefficient increase leads to the instability areas decrease for the second principal parametric resonances;
3. The numerical results have a reasonable agreement with the approximate analytical results. It demonstrates that the instability areas with the nonhomogeneous boundary conditions almost are no different with the homogeneous boundary conditions. So, the assumption about the homogeneous boundary conditions in all available works on axially moving viscoelastic materials is reasonable.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Project No. 11672186, 11502147, 11602146, 11232009, 11572182), “Chen Guang” Project Supported by Shanghai Municipal Education Commission and Shanghai Education Development Foundation (No. 14CG57).

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