Energetics and Conserved Functional of Axially Moving Materials Undergoing Transverse Nonlinear Vibration

Li-Qun Chen
Department of Mechanics & Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, 200436, China

Jean W. Zu
Department of Mechanical & Industrial Engineering, University of Toronto, Toronto, Ontario, M5S 3G8, Canada

[DOI: 10.1115/1.1760557]

1 Introduction

Axially moving materials can represent many engineering devices such as power transmission belts, elevator cables, plastic films, magnetic tapes, paper sheets, textile fibers, band saws, aerial cable tramways, and crane hoist cables [1–3]. Energetics of axially moving materials is of considerable interest in the study of axially moving materials.

The total mechanical energy associated with axially moving materials is not constant when the materials travel between two supports. It is a fundamental feature of free transverse vibration of axially moving materials, while the total energy is constant for an undamped non-translating string or beam. Chubachi [4] first discussed periodicity of the energy transfer in an axially moving string. Miranker [5] analyzed energetics of an axially moving string, and derived an expression for the time rate of change of the string energy. Barakat [6] considered the energetics of an axially moving beam and found that energy flux through the supports can invalidate the linear theories of both the axially moving string and beam at sufficiently high transporting speed. Tabarrok, Leech and Kim [7] showed that the total energy of a travelling beam without tension is periodic in time. Wickert and Mote [8] pointed out that Miranker’s expression represents the local rate of change only because it neglected the energy flux at the supports, and they presented the temporal variation of the total energy related to the local rate of change through the application of the one-dimensional transport theorem. They also calculated the temporal variation of energy associated with modes of moving strings and beams. Renshaw [9] examined the change of the total mechanical energy of two prototypical winching problems, which provided strikingly different examples of energy flux at a fixed orifice of an axially moving system. Lee and Mote [10,11] presented a generalized treatment of energetics of translating continua, including axially moving strings and beams. They considered the case that there were nonconservative forces acting on two boundaries. Renshaw, Rahn, Wickert and Mote [12] examined the energy of axially moving strings and beams from both Lagrangian and Eulerian views. Their studies indicated that Lagrangian and Eulerian energy functionals are not conserved for axially moving continua. Zhu and Ni [13] investigated energetics of axially moving strings and beams with arbitrarily varying lengths.

Although both the Eulerian and Lagrangian functionals for the total mechanical energy of axially moving materials are generally not constant, there do exist alternative functionals that are conserved. Miranker [5] actually found an Eulerian conserved functional for axially moving strings. Renshaw, Rahn, Wickert and Mote [8] presented both Eulerian and Lagrangian conserved functionals for the prototypical axially moving string and beam models. They concluded that the conserved Eulerian functionals qualify as Lyapunov functionals in stability analysis, while the conserved Lagrangian functionals do not qualify as Lyapunov functionals because their time derivatives are only defined at an instant.

Both energy and conserved functionals are potentially useful for stability analysis [14] and controller design [15]. Besides, these functionals can be used to develop and to validate numerical algorithms. However, all aforementioned research on energy and conserved functionals for axially moving materials is confined to linear models. There is very little literature that is specially related to energy and conserved functionals in nonlinear vibrations of axially moving materials. To address the lack of research in this aspect, the authors investigate energy and conserved functionals for axially moving strings and beams with geometric nonlinearity. In addition, while available research on energy functionals for axially moving materials has been focused on homogeneous boundary conditions such as the string with fixed ends and the clamped or the simply supported beams, the present investigation extends the work to treat non-homogeneous boundary conditions.

In the Technical Brief, the known energy and conserved functionals for linear vibration are extended to nonlinear vibration. The authors adopt the simplest models of axially moving materials undergoing nonlinear transverse vibration. The models are based on the assumptions that the transverse motion is in a plane, is not coupled with the longitudinal motion, and is small but not infinitesimal so that only the lowest order nonlinear term need to be retained in the governing equation. The Lagrangian strain is adopted to account for geometric nonlinearity due to small but finite stretching of the string. The Eulerian and Lagrangian energy functionals are defined in nonlinear vibration of axially moving strings and beams. Their time-rates of change are calculated under non-homogeneous boundary conditions. The conserved functionals in Eulerian sense are defined under some homogeneous boundary conditions for axially moving nonlinear strings and beams.

2 Energetics in Nonlinear Vibration of Axially Moving Strings

Consider a uniform axially moving string of linear density \( \rho \), cross-sectional area \( A \), Young’s modulus \( E \) and initial tension \( P \). The string travels at the constant and uniform axial transport speed \( V \) between two boundaries separated by distance \( L \). The distance from the left boundary is measured by fixed axial coordinate \( x \), and time is denoted by \( t \). The transverse displacement of the string is given by the Eulerian variable \( u(x,t) \) in the sense that \( u(x,t) \) describes the displacement of the string element instantaneously located at \( x \) even though different material elements occupy that position at different times. No out-of-plane motions are considered. Assuming the spatial derivatives of longitudinal displacement enough small to be ignored, Mote [16] derived the governing equation of nonlinear free vibration of the axially moving string as

\[
p du^2 + 2 \rho V u_x u_{xx} + \rho V^2 - P \frac{3}{2} E A u_x^2 \left. u_{xx} \right| = 0 \tag{1}
\]

where a comma denotes partial differentiation.

From the Eulerian view, consider the total mechanical energy in a specified spatial domain, the span \((0,L)\). Hence the Eulerian functional for the total mechanical energy consists of the kinetic energy of all material particles and the potential energy due to the initial tension and the disturbed tension caused by the transverse motion.
\[ E_c[u] = \int_0^L \left( \frac{\rho}{2} \left[ V^2 + (u_x + V u_x)^2 \right] + \left( P + \frac{1}{2} \sigma A \right) e \right) \, dx \]  

(2)

where \( u_x + V u_x \) is the absolute velocity of the material particle at \( x \) in the transverse direction and includes both local and convective components, and \( \sigma \) is the disturbed stress that is connected with corresponding Lagrangian strain

\[ e = \frac{1}{2} u_x^2 \]  

(3)

by the linear elastic constitutive relation

\[ \sigma = E e \]  

(4)

Here the Lagrangian strain is used to account for geometric non-linearity due to small but finite stretching of the string. The time rate of change of the Eulerian energy functional is

\[ \frac{dE_c[u]}{dt} = \int_0^L \frac{\partial}{\partial t} \left[ \frac{\rho}{2} \left[ V^2 + (u_x + V u_x)^2 \right] + \left( P + \frac{1}{2} \sigma A \right) e \right] \, dx \]  

(5)

Here the order of differentiation and integration can be interchanged as the limit of integration is time-independent. After some mathematical manipulations, one can cast Eq. (5) into the form

\[ \frac{dE_c[u]}{dt} = \int_0^L \left[ \left( u_x + V u_x \right) \rho u_{xx} + 2 \rho V u_{xt} + \left( \rho V^2 - P \right) u_x - \frac{3}{2} E A u_x^3 \right] \, dx \]  

(6)

From the governing Eq. (1), one gets the time-rate of change

\[ \frac{dE_c[u]}{dt} = \left[ \frac{1}{2} V \left( P - \rho V^2 \right) u_x^2 + \frac{3}{8} V E A u_x^4 \right] + \left[ \rho V^2 - P \right] u_x \bigg|_0^L \]  

(7)

Equation (7) holds for non-homogeneous boundary conditions that stand for the ends moving in a prescribed way. If the string constrained by homogeneous boundary conditions of the motionless end,

\[ u(0,t) = u(L,t) = 0 \]  

(8)

Equation (7) reduces to a simple form

\[ \frac{dE_c[u]}{dt} = \left[ \frac{1}{2} V \left( P - \rho V^2 \right) u_x^2 + \frac{3}{8} V E A u_x^4 \right] \bigg|_0^L \]  

(9)

Obviously, the corresponding equation for linear vibration [12] is a special case of Eq. (9) when the high order term is omitted.

From the Lagrangian view, consider the total mechanical energy in a specified set of material particles between \( Vt \) and \( Vt + L \). Hence the Lagrangian functional for the total mechanical energy is

\[ E_L[u] = \int_{Vt}^{Vt+L} \left[ \frac{\rho}{2} \left[ V^2 + (u_x + V u_x)^2 \right] + \left( P + \frac{1}{2} \sigma A \right) e \right] \, dx \]  

(10)

which is defined only at \( t = 0 \). The time rate of change of the Lagrangian energy functional is derived from Leibnitz’s chain rule for integration with time-dependent limits

\[ \frac{dE_L[u]}{dt} = \int_{Vt}^{Vt+L} \frac{\partial}{\partial t} \left[ \frac{\rho}{2} \left[ V^2 + (u_x + V u_x)^2 \right] + \left( P + \frac{1}{2} \sigma A \right) e \right] \, dx \]  

(11)

Equation (11) can also be obtained through application of the one-dimensional transport theorem. Substituting Eq. (6) into Eq. (11) yields

\[ \frac{dE_L[u]}{dt} = \int_{Vt}^{Vt+L} \left[ PV u_x^2 + \frac{1}{2} V E A u_x^4 + u_x \left( Pu_{xx} + \frac{1}{2} E A u_x^4 \right) \right] \, dx \]  

(12)

Equations (11) and (12) are only valid at \( t = 0 \) because \( E_L[u] \) is only defined by Eq. (10) at \( t = 0 \) and the governing Eq. (1) holds only when the material particles associated with \( E_L[u] \) comprise the span. From Eqs. (7) and (12), \( dE_L[u]/dt \) and \( dE_c[u]/dt \) are distinct although \( E_L[u] = E_c[u] \) at \( t = 0 \). Equation (12) holds for non-homogeneous boundary conditions.

If the string constrained by homogeneous boundary conditions (8), Eq. (12) leads to

\[ \frac{dE_L[u]}{dt} = \left[ PV u_x^2 + \frac{1}{2} E A u_x^4 \right] \bigg|_0^L \]  

(13)

Neglecting the higher order term in Eq. (13) gives the corresponding result in linear vibration [8,12].

### 3 Energetics in Nonlinear Vibration of Axially Moving Beams

Consider a uniform axially moving beam of linear density \( \rho \), cross-sectional area \( A \), cross-sectional area moment of inertia \( I \), Young’s modulus \( E \) and initial tension \( P \). The beam travels at the constant and uniform axial transport speed \( V \) between two boundaries separated by distance \( L \). The Eulerian variable \( u(x,t) \) stands for the transverse displacement of the beam element instantaneously located at fixed axial coordinate \( x \) and at time \( t \). Thurman and Mote [17] presented the governing equation of planar transverse free nonlinear vibration of the axially moving beam

\[ \rho u_{tt} + 2 \rho V u_{xt} + \left( \rho V^2 - P - \frac{3}{2} E A u_x^2 \right) u_{xx} + E I u_{xxxx} = 0 \]  

(14)

The Eulerian functional for the total mechanical energy of spatial domain \((0,L)\), consisting of the kinetic energy of all material particles and the potential energy due to the initial tension, the disturbed tension, and the bending moment, is defined by

\[ E_L[u] = \int_0^L \left[ \frac{1}{2} \left( V^2 + (u_x + V u_x)^2 \right) + \left( P + \frac{1}{2} \sigma A \right) e \right] \, dx \]  

(15)

where \( \sigma \) is the disturbed stress that is connected with the corresponding Lagrangian strain in Eq. (3) by the linear elastic constitutive relation, Eq. (4). The time rate of change of the Eulerian energy functional is

\[ \frac{dE_L[u]}{dt} = \int_0^L \frac{\partial}{\partial t} \left[ \frac{1}{2} \left( V^2 + (u_x + V u_x)^2 \right) + \left( P + \frac{1}{2} \sigma A \right) e \right] \, dx \]  

(16)
Using the governing Eq. (18), one can simplify Eq. (20) and obtain the time-rate of change
\[
\frac{dE_L[u]}{dt} = \left[ \frac{1}{2} V(P - \rho V^2)u_x^2 + \frac{3}{8} V E A u_x^4 + \frac{1}{2} V E I u_{x x x x} \right]_0^L
- V E I u_{x x x} u_x + u_x \left( P - \rho V^2 \right) u_x + \frac{1}{2} V E A u_x^3 - \frac{1}{2} \rho V u_x \left. \right|_0^L
- E I u_{x x x} + E I u_{x x x} u_x \right|_0^L
\] (17)

Equation (17) holds for non-homogeneous boundary conditions.

For the simply supported axially moving beam, the boundary conditions are from Eq. (8) and
\[
u_{x x}(0, t) = u_{x x}(L, t) = 0
\] (18)

From Eqs. (17), (8) and (18)
\[
\frac{dE_L[u]}{dt} = \left[ \frac{1}{2} V(P - \rho V^2)u_x^2 + \frac{3}{8} V E A u_x^4 - E I u_{x x x x} u_x \right]_0^L
\] (19)

is the time-rate of change of the Eulerian energy functional for the simply supported axially moving beam. Comparison with the corresponding result for linear vibration [12] if the high order term is omitted.

For the clamped axially moving beam, the boundary conditions are Eq. (8) and
\[
u_{x x}(0, t) = u_{x x}(L, t) = 0
\] (20)

From Eqs. (17), (8) and (20)
\[
\frac{dE_L[u]}{dt} = \frac{1}{2} V E I u_{x x x x}^2 \right|_0^L
\] (21)

is the time-rate of change of the Eulerian energy functional for the clamped axially moving beam. Comparing with the corresponding result in [12], one knows that the nonlinear term does not affect the time-rate of change.

The Lagrangian functional for the total mechanical energy of all material particles between \(Vt\) and \(Vt + L\) is defined by
\[
E_L[u] = \int_{Vt}^{Vt+L} \left[ \frac{1}{2} \rho \left( V^2 + (u_x + V u_x)^2 \right) \right. \nonumber
+ \left. \frac{1}{2} P u_x^2 + \frac{3}{8} V E A u_x^4 + \frac{1}{2} V E I u_{x x x x} \right] dx
\] (22)
at \(t = 0\). Its time rate of change can be derived from the governing equation and the Leibniz chair rule as
\[
\frac{dE_L[u]}{dt} = \left[ PV u_x^2 + \frac{1}{2} V E A u_x^4 + E I u_{x x x x} u_x + u_x \left( P - \rho V^2 \right) u_x + \frac{1}{2} V E A u_x^3 \right]_0^L
- E I u_{x x x} + E I u_{x x x} u_x \right|_0^L
\] (23)

Equation (23) holds for non-homogeneous boundary conditions.

For the simply supported axially moving beam, the boundary conditions are Eqs. (8) and (18). Hence
\[
\frac{dE_L[u]}{dt} = \left[ PV u_x^2 + \frac{1}{2} V E A u_x^4 - E I u_{x x x x} u_x \right]_0^L
\] (24)
is the time-rate of change of the Lagrangian energy functional for the simply supported axially moving beam. The corresponding result for linear vibration [8,12] can be derived from Eq. (24) through neglecting high order term.

For the clamped axially moving beam, the boundary conditions are Eq. (8) and (20). Thus
\[
\frac{dE_L[u]}{dt} = V E I u_{x x x x}^2 \right|_0^L
\] (25)

the time-rate of change of the Lagrangian energy functional for the clamped axially moving beam. Comparison with the corresponding result in [8,12] indicates that the nonlinear effect does not appear in the time rate of change.

4 Conserved Functional for Axially Moving Materials

Generally both the Eulerian and Lagrangian energy functionals are not constant in nonlinear vibration of axially moving materials. However, under specific boundary conditions, there do exist alternative functionals that are conserved. As conserved Lagrangian functionals cannot serve as Lyapunov functionals, this paper discusses only the conserved Eulerian functionals.

For nonlinear transverse vibration of the axially moving string, governed by Eq. (1) and constrained by condition (8), the Eulerian functional
\[
S_E[u] = \int_0^L \frac{1}{2} \rho u_x^2 + \frac{3}{8} (P - \rho V^2) u_x^3 + \frac{1}{8} E A u_x^4 dx
\] (26)
is conserved. In fact, calculating the temporal differentiation
\[
\frac{dS_E[u]}{dt} = \int_0^L \frac{\partial}{\partial t} \left[ \frac{1}{2} \rho u_x^2 + \frac{3}{8} (P - \rho V^2) u_x^3 + \frac{1}{8} E A u_x^4 \right] u_t \right|_0^L
\] (27)
one obtains
\[
\frac{dS_E[u]}{dt} = \int_0^L u_t \left[ \rho u_{t x} u_x + \frac{3}{8} (P - \rho V^2) u_x^3 \right] dx
\] (28)

Therefore Eqs. (1) and (8) result in \(dS_E[u]/dt = 0\). The conserved Eulerian functional (26) reduces the same functional to that for linear vibration [12]. Similar to the linear case, the conserved Eulerian functional \(S_E[u]\) is the Jacobi integrals of the axially moving string. However, in nonlinear case, \(S_E[u]\) is positive definite for the critical speed of the moving string, \(V = \sqrt{\rho/\rho}\), while is positive definite for \(V < \sqrt{\rho/\rho}\) in linear case [12].

For nonlinear transverse vibration of the simply supported or clamped axially moving beam, governed by Eq. (14) and constrained by conditions (8) and (18) or (20), the Eulerian functional
\[
B_E[u] = \int_0^L \frac{1}{2} \rho u_x^2 + \frac{3}{8} (P - \rho V^2) u_x^3 + \frac{1}{8} E A u_x^4 + \frac{1}{2} E I u_{x x x x}^2 \right|_0^L
\] (29)
is conserved, because
\[
\frac{dB_E[u]}{dt} = \int_0^L \rho u_{t x} u_x + \frac{3}{8} (P - \rho V^2) u_x^3 \right|_0^L
+ E I u_{x x x x} u_x \right|_0^L
\] (30)

which is zero by inserting Eqs. (14), (8) and (18) or (20).

It should be pointed out that the Eulerian functionals \(S_E[u]\) and \(B_E[u]\) are not conserved for non-homogeneous boundary conditions. \(S_E[u]\) is a conserved functional only for an axially moving string with two fixed ends. \(B_E[u]\) is a conserved functional only for an axially moving beam with two simply supported ends, two clamped ends, or a simply supported end and a clamped end. When the high order term is neglected, Eqs. (27) and (29) lead to
the Eulerian conserved functionals in linear vibrations of axially moving strings and axially moving beams, respectively [12].

5 Summary

In this paper, the authors define the Eulerian and Lagrangian energy functionals for axially moving strings and beams undergoing transverse nonlinear vibration. The Lagrangian strain is used to account for geometric nonlinearity due to small but finite stretching of the string or the beam. The time rates of change are calculated for energy functionals of both axially moving strings and beams. The Eulerian view is concerned with a specified spatial domain while the Lagrangian view is concerned with a specified set of particles. The rate of change of the Eulerian energy functional accounts for the net flux of energy at the two boundaries, and that of the Lagrangian energy functional equals the rate of work done by the boundary forces and/or moments. Therefore the time rates of change of the Eulerian and Lagrangian energy functionals are distinct. The time rates of change of energy functionals are determined for boundaries transversely moving in an arbitrarily given way, which include strings with fixed ends, beams with simply supported or clamped ends as special cases. Their time rates of change indicate that all energy functionals are not conserved during the process of vibration. This paper also presents the Eulerian conserved functionals for axially moving nonlinear strings and beams under specific boundary conditions. All functionals defined here and their time rate of change calculated reduce to the corresponding known results of linear vibration of axially moving materials if the nonlinear term is dropped out. In present investigation, the simplest governing equations are used to model nonlinear transverse vibration of axially moving materials. Energetics and conserved functionals can be studied based on more sophisticated models to account some factors ignored here. Physically, the rate of energy change in the Eulerian or Lagrangian view can be expressed by the boundary values, and conserved functionals can be found, but the mathematical derivation will be much more complicated.

Acknowledgment

The research is supported by the National Natural Science Foundation of China (Project No. 10172056).

References