The parametric open-plus-closed-loop control of chaotic maps and its robustness

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Abstract

This paper proposes a parametric open-plus-closed-loop control approach to controlling chaos. The logistic map is treated as an example to demonstrate the application of the proposed approach. It is proved that the approach is robust to the model error. Its relations to the open-plus-closed-loop control and the parametric entrainment control are discussed.

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1. Introduction

Over the past decade, there has been a great deal of research related to the control and synchronization of chaos [1–3]. Jackson and Grosu advanced a new powerful method of controlling chaos, the open-plus-closed-loop method, for dynamical systems defined by a set of first-order ordinary differential equations [4]. This method may be implemented in experiments [5–8]. The idea has been employed to control chaotic oscillators defined by a set of non-autonomous second-order ordinary differential equation [9], and discrete dynamical systems defined by a set of difference equations [10]. The idea has also been applied in synchronization of chaotic maps [11,12]. The open-plus-closed-loop control can be only applied to the systems with additive controllable parameters. For systems without additive controllable parameters, the parametric entrainment control [13] can be developed to the parametric open-plus-closed-loop control. Actually, the parametric open-plus-closed-loop control has been proposed for general continuous-time dynamical systems defined by a set of first-order ordinary differential equations [14] and oscillating systems defined by a set of non-autonomous second-order ordinary differential equation [15]. This paper presents the parametric open-plus-closed-loop control for discrete-time dynamical systems defined by maps. Recently, the attention has also been paid to the robustness and the adaptiveness of approaches to control chaos [16–18]. This paper demonstrates that the parametric open-plus-closed-loop control is robust to the model error.

2. The parametric open-plus-closed-loop control

Consider a map

\[ x_{n+1} = f(x_n, u_n) \]  (1)
where \( x_n \in \mathbb{R}^m \) and \( u_n \in \mathbb{R}^m \) denote respectively the state variable and the control input of the system at time step \( n \). Suppose that \( f' (j = 1, 2, \ldots, m) \), components of \( f \), be twice differentiable with respect to \( x_n' \) and \( u_n' \) (\( i = 1, 2, \ldots, m \)), components of \( x_n \) and \( u_n \), respectively. The partial derivative with respect to \( u_n' \), calculated at \( (g_n, 0) \) vanishes, where \( g_n \) is the given control goal. Then \( Df_n = \frac{\partial f}{\partial u_n} \|_{(g_n, 0)} \) does not involve \( u_n' \) explicitly. Suppose \( Df_n \) be invertible. Denote \( Df_n = \frac{\partial f}{\partial x_n} \|_{(g_n, 0)} \). Given a small positive number \( \delta \), supposing the control be activated after time step \( n_0 \) and denoting switch functions

\[
S_1(n) = \begin{cases} 
0 & n < n_0 \\
1 & n \geq n_0 
\end{cases} \\
S_2(u) = \begin{cases} 
1 & z < \delta \\
0 & z \geq \delta 
\end{cases}
\]

and \( A_n \) be a \( m \times m \) matrix sequence whose magnitudes of \( m \) eigenvalues are all less than 1 at all steps, then one has the parametric open-plus-closed-loop control law

\[
u_n = S_1(n)S_2(|x_n - g_n|)(Df_n)^{-1}[g_{n+1} - f(g_n, 0) + (Df_n - A_n)(g_n - x_n)]
\]

such that the dynamical behavior of system (1) are entrained to the goal \( g_n \), that is,

\[
\lim_{n \to \infty} |x_n - g_n| = 0
\]

Denote

\[
B(g) = \{x_n | n > n_0, |x_n - g_n| < \delta\}
\]

If \( B(g) \) is not empty, it is easy to prove the convergence of the control law (3). In fact, choosing the lest \( n \) such that \( x_n \in B(g) \) as \( n \) and linearizing locally Eq. (1) at \( (g_n, 0) \)

\[
x_{n+1} = f(g_n, 0) + Df_n(x_n - g_n) + Df_n(u_n - 0)
\]

if \( n > n_1 \), from Eq. (3) and Eq. (6), one has

\[
x_{n+1} - g_{n+1} = A_n(x_n - g_n)
\]

As the magnitudes of all eigenvalues of \( A_n \) are less than 1, Eq. (4) is proved.

In practical applications, \( A_n \) may be a constant matrix \( A \). The parametric open-plus-closed-loop control can only be employed after \( x_n \) comes into \( B(g) \). For chaotic systems, the technique of targeting [19] may be utilized to shorten the waiting time. Therefore the parametric open-plus-closed-loop control is not restricted to, but especially fitted for controlling chaos.

Now the logistic map with an unadditive controllable parameter

\[
x_{n+1} = (4 + u_n)x_n(1 - x_n) \quad x_n \in [0, 1], \quad u_n \in \mathbb{R}
\]

is treated as an example to demonstrate the application of the parametric open-plus-closed-loop control. Chaotic behavior appears in system (8) if no control \( (u_n = 0) \). The control goals successively are

\[
g_{n+1} = 3.4g_n(1 - g_n)
\]

\[
g_{n+1} = 3.5g_n(1 - g_n)
\]

whose steady-state behaviors are a period-2 orbit and a period-4 orbit respectively. Start control at \( n_0 = 150 \) steps and let \( \delta = 0.02 \) and \( A = 0.5 \). According to Eq. (3),

\[
u_n = S_1(n)S_2(|x_n - g_n|)[g_{n+1} - 4g_n(1 - g_n) + (1 - 2g_n)(g_n - x_n)]
\]

where

\[
S_1(n) = \begin{cases} 
0 & n < 150 \\
1 & n \geq 150 
\end{cases} \\
S_2(|x_n - g_n|) = \begin{cases} 
1 & |x_n - g_n| < 0.02 \\
0 & |x_n - g_n| \geq 0.02
\end{cases}
\]

The results are shown in Figs. 1 and 2 in which the controls are ended at 250 steps.

It should be remarked that known controlling chaos approaches, except those directly adopted from control engineering [3], fail to control map (8) to the goals (9) and (10). The entrainment control and the open-plus-closed-loop control cannot be applied since map (8) has no additive controllable parameter. The parametric entrainment control can be applied in principle, but fail to be in practice because the goals (9) and (10) are not in the convergent region of the
system. The OGY approach and its variations such as the occasional proportional feedback control [1–3] can stabilize one of the unstable periodic orbits embedded in its chaotic attractor, but they cannot control map (8) to the goals (9) and (10) as the steady-state behaviors of maps (9) and (10) are not the unstable periodic orbits of map (8).

3. Robustness of the parametric open-plus-closed-loop control

Assume that system (1) be only a sufficiently good mathematical model of the real experimental system described by the map

\[ x_{n+1} = E(x_n, u_n) \]

The robustness here means that the real system (13) may be controlled sufficiently well by the control law (3) designed for the model system (1) if the model error is small enough. Suppose that the model system (1) and the real system (13) vary with control parameters \( u_n \) in the same way, that is

\[ D^2 E_n = \frac{\partial^2 E_n}{\partial u_n^2} = D^2 f_n \]

Those satisfying condition (14) encompass the special cases of the system with additive control parameters and the system that the difference between the experiment and model systems is expressed in different parameter vectors [13]. Let distance in the Euclidean space \( \| \bullet \| \) be used as the norm in \( \mathbb{R}^n \), and induced matrix norm as \( \| \bullet \| \). For a sequence \( y_n \), denote the model error as

\[
\max |E(y_n, 0) - f(y_n, 0)| = \varepsilon
\]

\[
\max \left\| \frac{\partial E}{\partial x_n} \right\|_{(y_n, 0)} - \left\| \frac{\partial f}{\partial x_n} \right\|_{(y_n, 0)} = \eta
\]
If the model is sufficiently good, \( e \) and \( \eta \) are small. Since the all magnitudes of \( m \) eigenvalues are less than 1, \( \|A_n\| < 1 \). Choose the matrix sequence \( A_n \) such that
\[
\|A_n\| = \|A\| = 1 - \eta
\]

Linearizing locally Eq. (13) at \( (g_n, 0) \), one has
\[
x_{n+1} = E(g_n, 0) + D_1E_n(x_n - g_n) + D_2E_n(u_n - 0)
\]
where
\[
D_1E_n = \left[ \frac{\partial F}{\partial x_n} \right]_{(g_n, 0)}
\]
Let
\[
A_n = |x_n - g_n|
\]
From Eqs. (17), (3), (14), (15) and (16), if \( n > n_1 \)
\[
A_{n+1} = |E(g_n, 0) - f(g_n, 0)| + (D_1E_n - D_1f_n)(x_n - g_n) + A_n |x_n - g_n| - \|E(g_n, 0) - f(g_n, 0)| + \|D_1E_n - D_1f_n||x_n - g_n| + \|A_n\||x_n - g_n|
\]
\[
< e + (\eta + \|A\|)A_n
\]
Therefore
\[
A_n < e \sum_{i=0}^{n-n_1} (\eta + \|A\|)^\gamma (\eta + \|A\|)^{n-n_1} \delta = e \frac{1}{(\eta + \|A\|)(\eta + \|A\|)^{n-n_1}} (1 - \eta - \|A\|) + (\eta + \|A\|)^{n-n_1} \delta
\]
Letting \( n \to \infty \) in Eq. (21) and noticing Eq. (16), one gets
\[
\lim_{n \to \infty} |x_n - g_n| < \frac{e}{1 - \eta - \|A\|}
\]
which indicates that one can design the parametric open-plus-closed-loop control law from the model system (1) to achieve good control results for the real system (13) when the model error \( e \) and \( \eta \) are small. Thus the parametric open-plus-closed-loop control is robust.

4. Discussions

Without loss of generality, the author only considers the case that \( n > n_1 \). Denoting
\[
A_n = Df_n
\]
in Eq. (3), one obtains the first approximation of the parametric entrainment control law [13]
\[
u_n = (D_2f_n)^{-1}[g_{n+1} - f(g_n, 0)]
\]
If the controllable parameters in system (1) are additive, Eq. (1) can be rewritten as
\[
x_{n+1} = F(x_n) + u_n, \quad x_n, u_n \in \mathbb{R}^m
\]
In this case, \( D_2f_n \) is the identity matrix. Thus Eq. (3) gives the open-plus-closed-loop control law [4]
\[
u_n = g_{n+1} - F(g_n) - \left( \frac{\partial F}{\partial x_n} \right)_{g_n} (g_n - x_n)
\]
In Eq. (26), choosing
\[
A_n = \left[ \frac{\partial F}{\partial x_n} \right]_{g_n}
\]
one has the entrainment control law [20]
\[ u_n = g_{n+1} - f \left( g_n \right) \] (28)

The (parametric) open-plus-closed-loop control requires that the magnitudes of eigenvalues of \( A_n \) be less than 1, which, according to Eqs. (23) and (27), is assured by the existence of convergent regions in the (parametric) entrainment control that is open-loop. Introducing the closed-loop makes (parametric) open-plus-closed-loop control be applicable in conservative systems without attractors. The (parametric) open-plus-closed-loop control still depends on initial conditions, but its suitability of initial conditions is more definite than that of the basin of entrainment for the (parametric) entrainment control. It requires that \( B(g) \) be not empty. In addition, if \( g_n \) is an unstable periodic orbit of the uncontrolled system, that is,
\[ g_{n+1} = f \left( g_n, 0 \right) \] (29)

the (parametric) entrainment control cannot be applied, but, from Eq. (3), one knows that
\[ u_n = \left( D_2 f_n \right)^{-1} \left( D_1 f_n - A_n \right) \left( g_n - x_n \right) \] (30)
can stabilize \( g_n \). In this case, the (parametric) open-plus-closed-loop control is essentially a local linear feedback control, while the controller is time-varying instead of time-independent. Therefore, the main idea of the (parametric) open-plus-closed-loop control is that the open-loop part creates a desired orbit and the closed-loop part stabilizes it. Correspondingly, the (parametric) entrainment control only creates a desired orbit whose stability is assured by the existence of convergent regions and the basin of entrainment.

Similar to the application of the open-plus-closed-loop control in migration problems [8], the parametric open-plus-closed-loop control can be applied to the migration among attractors of systems without additive parameters.

5. Conclusions

This paper presents the parametric open-plus-closed-loop approach for controlling chaos. The method is applied to control chaos in the logistic map to period-2 and period-4 orbits. The approach is proven to be robust to the model error.

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References