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Principal parametric resonance of axially accelerating viscoelastic strings with an integral constitutive law

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The steady-state transverse responses and the stability of an axially accelerating viscoelastic string are investigated. The governing equation is derived from the Eulerian equation of motion of a continuum, which leads to the Mote model for transverse motion. The Kirchhoff model is derived from the Mote model by replacing the tension with the averaged tension over the string. The method of multiple scales is applied to the two models in the case of principal parametric resonance. Closed-form expressions of the amplitudes and the existence conditions of steady-state periodical responses are presented. The Lyapunov linearized stability theory is employed to demonstrate that the first (second) non-trivial steady-state response is always stable (unstable). Numerical calculations show that the two models are qualitatively the same, but quantitatively different. Numerical results are also presented to highlight the effects of the mean axial speed, the axial-speed fluctuation amplitude, and the viscoelastic parameters.

Keywords: nonlinear parametric vibration; method of multiple scales; stability; axially moving string; viscoelasticity

1. Introduction

Many engineering devices involve transverse vibrations of axially moving strings. Serpentine belts, fibre windings, magnetic tapes and thread lines all belong to this class. One important problem is the occurrence of large transverse vibrations, termed as ‘parametric vibration’, in such systems. Much research has addressed this issue, and has been reviewed by Wickert & Mote (1988), Abrate (1992), Chen & Zu (2001) and Chen (2005).

The parametric vibration is usually excited by the tension fluctuation or the axial acceleration. Although Miranker (1960) first derived the equation of transverse vibration for an axially accelerating string, detailed research had not been conducted until recently. Pakdemirli & Batan (1993) used the Galerkin method and Floquet theory to treat numerically the dynamic
stability of a constantly accelerating string. Pakdemirli et al. (1994) further considered the dynamic stability of a moving string where the time dependent axial velocity is sinusoidal. In the case for which the time dependent axial velocity varies harmonically about a constant mean velocity, Pakdemirli & Ulsoy (1997) applied the discretization-perturbation and the direct-perturbation methods to analyse the stability of an axially accelerating string. For the arbitrary time dependent axial velocity, Ozkaya & Pakdemirli (2000) employed the Lie group theory to find exact solutions. All of these works are confined to linear models. Chung et al. (2001) applied the generalized-α method to study numerically the coupled transverse and longitudinal motion of an axially accelerating elastic string with geometric nonlinearity. Chen & Zhao (2005) proposed a numerical algorithm to simulate nonlinear transverse vibration of an axially accelerating viscoelastic string by discretizing the governing equation base on a set of independent functions. Chen et al. (2004b) applied the Galerkin method to study numerically the long-time behaviours in transverse vibration of an axially accelerating viscoelastic string. Chen et al. (2004a) employed the method of multiple scales to derive the closed-form expressions of the amplitudes of steady-state periodical motion and their stability conditions in principal parametric resonance of axially accelerating strings. In these works, the Kelvin relation is used to characterize the viscoelasticity of the strings. Compared with the Kelvin relation, the integral types of viscoelastic constitutive laws, such as the Boltzmann superposition principle, are more widely adopted in recent years because they are able to represent more complicated time-dependent material properties. For strings constituted by the Boltzmann superposition principle, Fung et al. (1997) used the Galerkin method based on stationary string eigenfunctions to compute transverse vibration of an axially accelerating string. Zhang et al. (2002) studied the same problem based on translating string eigenfunctions. Chen et al. (2004c) applied the Galerkin method to investigate numerically the parametric vibration of an axially accelerating string constituted by a fractional differentiation law. There is no analytical investigation on transverse vibrations of axially accelerating strings constituted by the viscoelastic constitutive law of an integral type. To address this lack of research, the authors investigate principal parametric resonance of an axially accelerating string constituted by the Boltzmann superposition principle.

To formulate the problem in a general framework, this paper develops the governing equation for the in-plane motion of axially accelerating strings, and then reduces it into the transverse motion case. Most researchers derived the governing equations of axially moving strings from Hamilton’s principle. Among other variational principles, Hamilton’s principle is formulated on the stress–strain relations (Washizu 1982). Variational principles are powerful tools to model elastic continua, but they are usually inappropriate for developing governing equations of viscoelastic continua, although there are some variational theorems derived from the governing equations of viscoelastic continua (Christensen 1982). In this paper, the Eulerian equation of motion is used to develop the governing equation of an axially accelerating viscoelastic string. When only the transverse motion is considered, the governing equation is reduced into the Mote model. In addition, this paper demonstrates that the
Kirchhoff model is an approximation of the Mote model by substituting the exact string tension with its average over the string.

Among several perturbation methods (Nayfeh & Mook 1979; Andrianov et al. 2003; Cartmell et al. 2003), the method of multiple scales is a powerful tool to seek approximate analytical solutions to nonlinear differential equations. Pakdemirli & Ulsoy (1997) and Zhang & Zu (1999), respectively, applied the method to investigate linear and nonlinear parametric vibration of axially moving strings without discretization of the governing equations. Their investigations are limited to elastic strings or viscoelastic strings constituted by the Kelvin relation. Chen & Zu (2003) and Chen et al. (2003) developed the method of multiple scales for a nonlinear integro-partial differential equation to study the principal resonance and summation resonance of axially moving strings constituted by the Boltzmann superposition principle, respectively. The parametric excitation is the tension fluctuation instead of axial-speed variation. However, the approach used there is only suitable for the stress relaxation function with small viscoelastic exponent. In the present investigation, the stress is treated as a new auxiliary unknown, and then the integro-partial differential equation is transformed into a set of two partial differential equations, which can be analysed by the method of multiple scales.

2. Two dynamical models of transverse motions

Consider an axially moving string of density $\rho$, area of cross-section $A$, initial tension $P_0$ and uniform transport speed $\gamma(t)$ that is a prescribed function of time $t$. Assume that the deformation of the string is confined to the vertical plane. The string is subjected to no external loads. A mixed Eulerian–Lagrangian description (Koivurova & Salonen 1999) is adopted. The in-plane vibration of the string is specified by $u(x,t)$ at axial spatial coordinate $x$ and time $t$, the longitudinal displacement related to coordinated translating at speed $\gamma(t)$, and $v(x,t)$, the transverse displacement related to a spatial frame. The physical system is shown in figure 1.
The string is a one-dimensional continuum undergoing an in-plane motion, and the Eulerian equation of motion of a continuum (Fung 1965) gives

$$\rho \frac{D}{Dt} \left( \gamma + \frac{Du}{Dt} \right) = \frac{\partial}{\partial x} \left( \frac{P_u}{A} \right), \quad \rho \frac{D^2v}{Dt^2} = \frac{\partial}{\partial x} \left( \frac{P_v}{A} \right), \quad (2.1)$$

where $P_u$ and $P_v$, respectively, are longitudinal and transverse components of the tension in the string, and the material time derivative operator (Fung 1965) is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x}. \quad (2.2)$$

The disturbed stress of the string is denoted by $\sigma(x,t)$. An element of initial length $ds$ is deformed into $\sqrt{(1 + u_x)^2 + v_x^2} ds$ by the longitudinal and transverse displacements (Thurman & Mote 1969). Therefore, longitudinal and transverse components of the tension in the string are, respectively,

$$P_u = \frac{(P_0 + A\sigma)(1 + u_x)}{\sqrt{(1 + u_x)^2 + v_x^2}}, \quad P_v = \frac{(P_0 + A\sigma)v_x}{\sqrt{(1 + u_x)^2 + v_x^2}}. \quad (2.3)$$

Substituting equations (2.2) and (2.3) into equation (2.1) yields

$$\rho A(u_{tt} + \dot{\gamma}(1 + u_x) + 2\gamma u_{xt} + \gamma^2 u_{xx}) - \frac{\partial}{\partial x} \left[ \frac{(P_0 + A\sigma)(1 + u_x)}{\sqrt{(1 + u_x)^2 + v_x^2}} \right] = 0, \quad (2.4)$$

$$\rho A(v_{tt} + \dot{\gamma}v_x + 2\gamma v_{xt} + \gamma^2 v_{xx}) - \frac{\partial}{\partial x} \left[ \frac{(P_0 + A\sigma)v_x}{\sqrt{(1 + u_x)^2 + v_x^2}} \right] = 0.$$  

The disturbed strain $\varepsilon_N(x,t)$ of the string is given by the nonlinear geometric relation (Thurman & Mote 1969)

$$\varepsilon_N = \sqrt{(1 + u_x)^2 + v_x^2} - 1. \quad (2.5)$$

In the present investigation, the viscoelastic string is constituted by an integral relation, the Boltzmann superposition principle,

$$\sigma(x,t) = \varepsilon_N(x,t) E(0) + \int_0^t \dot{E}(t - t') \varepsilon(x,t') dt', \quad (2.6)$$

where $E(t)$ is the stress relaxation function. Equation (2.4), with equations (2.5) and (2.6), is the governing equation of in-plane vibration of an axially accelerating string. Substitution of equations (2.5) and (2.6) into (2.4) leads to the governing equation of an axially accelerating viscoelastic string. In particular, if the axial speed is a constant and the string is subjected to no external load, then the result here reduces to the governing equation obtained by Thurman & Mote (1969) and Koivurova & Salonen (1999). However, both studies derived the governing equations of axially moving elastic strings from
Hamilton’s principle, which is usually inapplicable to developing governing equations of viscoelastic strings.

Although the transverse motion is generally coupled with the longitudinal motion, it is a weak small-amplitude motion. Therefore, for small but finite stretching problems, one can only consider the transverse vibration. In this case, only the lowest-order nonlinear terms need to be retained. Inserting Eq. (2.4) and (2.5), and then omitting higher-order nonlinear terms, gives the dynamical equation of transverse vibration

$$\rho A (v_{tt} + \gamma v_x + 2 \gamma v_{xt} + \gamma^2 v_{xx}) - P_0 v_{xx} - \frac{\partial}{\partial x} (A \sigma v_x) = 0, \quad (2.7)$$

and the Lagrangian strain

$$\varepsilon_L = \frac{1}{2} v_x^2. \quad (2.8)$$

Substitution of equations (2.6) and (2.8) into equation (2.7) leads to the governing equation of transverse vibration

$$\rho A (v_{tt} + \gamma v_x + 2 \gamma v_{xt} + \gamma^2 v_{xx}) - P_0 v_{xx}$$

$$- \frac{A}{2} \frac{\partial}{\partial x} \left\{ \left[ E(0) v_x^2 + \int_0^t \dot{E}(t - t') v_{xx}^2 dt' \right] v_x \right\} = 0. \quad (2.9)$$

Because Mote (1966) first obtained a special case of equation (2.9) for an elastic ($\dot{E} = 0$) string moving with a constant axial speed ($\gamma = 0$), equation (2.9) is called the ‘Mote model’ for the axially moving string constituted by the Boltzmann superposition principle. Fung et al. (1997) applied a dynamic equation of motion in terms of Trefftz stress components to obtain equation (2.9). Chen et al. (2003) applied Newton’s second law to obtain a special case of equation (2.9) without the axial acceleration ($\gamma = 0$).

Kirchhoff’s (1877) nonlinear model from an elastic string can be extended to the viscoelastic string. If the spatial variation of the tension is rather small, then one can use the averaged value of the disturbed tension $(1/L) \int_0^L A \sigma \, dx$ to replace the exact value $A \sigma$. In this case, equation (2.7) becomes

$$\rho A (v_{tt} + \gamma v_x + 2 \gamma v_{xt} + \gamma^2 v_{xx}) - P_0 v_{xx} - \frac{v_{xx}}{l} \int_0^L A \sigma \, dx = 0. \quad (2.10)$$

Substitution of equations (2.6) and (2.8) into equation (2.10) yields

$$\rho A (v_{tt} + \gamma v_x + 2 \gamma v_{xt} + \gamma^2 v_{xx}) - P_0 v_{xx}$$

$$- \frac{A v_{xx}}{2l} \int_0^L \left[ E(0) v_x^2 + \int_0^t \dot{E}(t - t') v_{xx}^2 dt' \right] dx = 0. \quad (2.11)$$

Equation (2.11) is the Kirchhoff model for the axially moving string constituted by the Boltzmann superposition principle. Equation (2.11) can be obtained through decoupling the governing equation for coupled longitudinal and transverse vibration under the quasi-static stretch assumption. In fact, Moon &
Wickert (1997) derived the model in such a way for the elastic string, whose \( E(t) \) is a constant. The assumption means that the dynamic tension component is a function of time alone. In a traditional derivation, equation (2.11) seems more exact than equation (2.9) because the former is the transverse equation of motion, in which the longitudinal displacement field is taken into account. However, the derivation here indicates that equation (2.9) can be reduced to equation (2.11), based on the replacement of the disturbed tension by its averaged value. Hence, the Kirchhoff model is an approximate model of transverse vibration (Chen & Zhao 2005b).

3. Multi-scale analysis of principal parametric resonance

The axial speed is characterized as a small simple harmonic variation about the constant mean speed:

\[
\gamma(t) = \gamma_0 + \gamma_1 \sin \omega t \quad (\gamma_0, \gamma_1 > 0).
\]  

(3.1)

The assumption has a physical meaning. For example, if the axially moving string models a belt on a pair of rotating pulleys, then the torsional oscillation of the pulleys will result in a small fluctuation in the axial speed of the belt. Substituting equation (3.1) into equation (2.9) and transforming the resulting equation into the dimensionless form yields

\[
\begin{align*}
\sigma_{rr} + 2(c + c_1 \sin \Omega r) \sigma_{r\xi} + \left( c^2 + \frac{c_1^2}{2} + 2c_1 \sin \Omega r + \frac{c_1^2}{2} \sin 2\Omega r - 1 \right) \sigma_{\xi\xi} \\
+ \Omega c_1 \sigma_{r\xi} \cos \Omega r = & \frac{1}{2} \frac{\partial}{\partial \xi} \left[ \varepsilon \xi(\xi, \tau) \sigma_{r\xi} \right],
\end{align*}
\]

(3.2)

where

\[
\xi(\xi, \tau) = D_0[\sigma_{r\xi}(\xi, \tau)]^2 + \int_0^\tau D_\tau(\tau - \tau')[\sigma_{r\xi}(\xi, \tau')]^2 d\tau',
\]

(3.3)

and

\[
\begin{align*}
\sigma &= \frac{v}{l}, \quad \xi = \frac{x}{l}, \quad \tau = \frac{t}{l} \sqrt{\frac{P_0}{\rho A}}, \\
\gamma_0 &= \frac{P_0}{\rho A}, \quad D_0 = \frac{E(0)A}{P_0}, \quad D(\tau) = \frac{E(t)A}{P_0}, \\
c &= \gamma_0 \sqrt{\frac{P_0}{\rho A}}, \quad c_1 = \gamma_1 \sqrt{\frac{P_0}{\rho A}}, \quad \Omega = \omega l \sqrt{\frac{P_0}{\rho A}}, \quad \varepsilon \xi(\xi, \tau) = \frac{A\sigma(x, t)}{P_0}.
\end{align*}
\]

(3.4)

Here, the external load is supposed to be zero. In equation (3.2), a small dimensionless parameter \( \varepsilon \) is employed as a book-keeping device, which implies that the disturbed internal force \( A\sigma(x, t) \) is much smaller than the initial tension \( P_0 \).

In equation (3.3), one assumes \( D_0 = 1 \). Then, the stress relaxation function of the standard linear solid model takes the form

\[
D(\tau) = a + (1 - a)e^{-a\tau}.
\]

(3.5)
Substituting equation (3.5) into equation (3.3), differentiating the resulting equation with respect to \( \tau \) and using equation (3.3) again yields

\[
\zeta_{,\tau}(\xi, \tau) = -\alpha \zeta(\xi, \tau) + \sigma_{,\tau}(\xi, \tau) + \frac{\alpha \alpha}{2} \left[ \sigma(\xi, \tau) \right]^2.
\]

(3.6)

For the stress relaxation function given by equation (3.5), two nonlinear, partial differential equations (3.2) and (3.6) consist of a closed set of equations that governs the transverse vibration of the axially moving string. They are equivalent to a partial-differential integral equation (3.2), along with equation (3.3), but are easier to tackle via perturbation approaches. Actually, the method of multiple scales can be applied to equations (3.2) and (3.3) for small \( \alpha \). Chen et al. (2003) and Chen & Zu (2003) studied a similar case in which the tension, instead of the axial speed, is time-dependent.

The mass, gyroscopic, and linear stiffness operators are introduced as follows

\[
M = I, \quad G = 2c \frac{\partial}{\partial \xi}, \quad K = (c^2 - 1) \frac{\partial^2}{\partial \xi^2},
\]

(3.7)

where operators \( M \) and \( K \) are symmetric, and \( G \) is skew-symmetric. Because the variation of the transport speed is small, one may let \( \gamma_1/\gamma_0 = c_1/c = \epsilon \delta \). Then, substituting equations (3.4) and (3.7) into equation (3.2), one obtains a continuous gyroscopic system with some small nonlinear terms and some small parameter excitation terms—namely,

\[
M \sigma_{,\tau} + G \sigma_{,\tau} + K \sigma = \epsilon \left\{ \frac{1}{2} \frac{\partial}{\partial \xi} \left[ \zeta(\xi, \tau) \sigma_{,\xi} \right] - \Omega \delta c \cos \Omega \tau \sigma_{,\xi} - 2\delta c \sin \Omega \tau \sigma_{,\xi} \right.
\]

\[
- \frac{\delta \epsilon^2}{2} \left( \delta \epsilon + 4 \sin \Omega \tau + \delta \epsilon \cos 2\Omega \tau \right) \sigma_{,\xi} \right\}.
\]

(3.8)

The method of multiple scales will be employed to solve equation (3.8) and (3.6). A second-order uniform approximation is sought in the form

\[
\sigma(\xi, \tau) = \sigma_0(\xi, T_0, T_1) + \epsilon \sigma_1(\xi, T_0, T_1) + O(\epsilon^2),
\]

(3.9)

and a first-order approximation for \( \zeta \) is assumed,

\[
\zeta(\xi, \tau) = \zeta_1(\xi, T_0, T_1) + O(\epsilon),
\]

(3.10)

where \( T_0 = \tau \) is a fast scale characterizing motion occurring at \( \omega_m \) (one of the natural frequencies of the corresponding unperturbed linear system), and \( T_1 = \epsilon \tau \) is a slow scale characterizing the modulation of the amplitudes and phases owing to the nonlinearity, viscoelasticity and possible resonance. Note that only the first-order approximation is kept in the expansion of \( \zeta \), because the right-hand side in equation (3.10) is already of the order of \( \epsilon \) and thus any higher-order expansion of \( \zeta \) will be unnecessary for the first approximate solutions.

Substituting equations (3.9) and (3.10) as well as the relationship

\[
\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + O(\epsilon^2)
\]

(3.11)
into equations (3.8) and (3.6), and equating the coefficients of like powers of $\epsilon$ in, the resulting equations finally yield

$$M\sigma_{0,T_0} + G\sigma_{0, T_0} + K\sigma_0 = 0, \quad (3.12)$$

$$M\sigma_{1,T_0} + G\sigma_{1, T_0} + K\sigma_1$$

$$= \frac{1}{2}[\zeta_1(\xi, \tau)]\sigma_{0,\xi}, \quad -2\delta c \sin \Omega T_0\sigma_{0,\xi T_0} - 2\delta c^2 \sin \Omega T_0\sigma_{0,\xi T_0}$$

$$- \Omega\delta c \cos \Omega T_0\sigma_{0,\xi} - 2M\sigma_{0,T_0}T_0 - G\sigma_{0, T_1}, \quad (3.13)$$

$$\alpha\zeta_1 + \frac{\partial\zeta_1}{\partial T_0} = \frac{\partial^2\sigma_{0}}{\partial\xi\partial T_0} - \frac{a\alpha}{2} \left( \frac{\partial\sigma_{0}}{\partial \xi} \right)^2. \quad (3.14)$$

Under the homogeneous boundary conditions, Wickert & Mote (1990) gave the solution to equation (3.12) as

$$\sigma_{0}(\xi, T_0, T_1) = \sum_{m=\pm 1, \ldots} \left[ \phi_m(\xi)A_m(T_1)e^{i\omega_m T_0} + \bar{\phi}_m(\xi)\bar{A}_m(T_1)e^{-i\omega_m T_0} \right], \quad (3.15)$$

where the overbar denotes complex conjugation, and the $m$th natural frequency and the $m$th normalized complex eigenfunction of the displacement field are, respectively, given by

$$\omega_m = m\pi(1 - c^2), \quad \phi_m(\xi) = \sqrt{2}\sin(m\pi\xi)e^{im\pi\xi}, \quad (3.16)$$

that satisfy the orthonormality relations

$$\langle \phi_m, M\phi_k \rangle = \delta_{mk}, \quad \langle \phi_m, G\phi_k \rangle = 2im\pi c^2\delta_{mk}. \quad (3.17)$$

Here, the inner product is defined as

$$\langle \phi_m, \phi_k \rangle = \int_0^1 \bar{\phi}_m\phi_k \, d\xi. \quad (3.18)$$

To investigate the principal parametric resonance, the solution of equation (3.12) can be expressed as

$$\sigma_0 = \phi_k(\xi)A_k(T_1)e^{i\omega_k T_0} + cc, \quad (3.19)$$

where $cc$ represents the complex conjugate of all preceding terms on the right-hand side of an equation.

Substituting equation (3.19) into equation (3.14) and solving the resulting equation yields

$$\zeta_1 = \frac{A_k^2 e^{i2\omega_k T_0}}{\alpha + i2\omega_k} \left( \frac{a\alpha}{2} + i\omega_k \right) (\phi_k')^2 + \frac{1}{2} aA_k\bar{A}_k\phi_k'\phi_k'' + cc, \quad (3.20)$$

where an overbar denotes a complex conjugate and a prime denotes differentiation with respect to $\xi$.

If the speed variation frequency $\Omega$ approaches twice a natural frequency of equation (3.12), then principal parametric resonance may occur. A detuning
parameter $\mu$ is introduced to quantify the deviation of $\Omega$ from $2\omega_k$, and $\Omega$ is described by

$$\Omega = 2\omega_k + \varepsilon \mu.$$  \hfill (3.21)

Based on equations (3.19), (3.20) and (3.21), equation (3.13) can be cast into

$$M\ddot{\phi}_k + G\ddot{\phi}_k + K\phi_k$$

$$= -\frac{dA_k}{dT_1} (2i\omega_k I\phi_k + M\phi_k') e^{i\omega T_0}$$

$$- \delta c \left[ \frac{\Omega}{2} - \omega_k \right] \phi_k' - i c \phi_k'' \right] \tilde{A}_k e^{i\omega T_0} e^{i\mu T_1}$$

$$+ \frac{1}{2} \left[ \frac{(2i\omega_k + a\alpha)}{4i\omega_k + 2\alpha} + a \right] A_k^2 \tilde{A}_k (2\phi_k' \phi_k'' + \phi_k'^2 \phi_k''') e^{i\omega T_0} + \text{NST} + \text{cc},$$ \hfill (3.22)

where NST stands for non-secular terms.

Equation (3.22) has a bounded solution only if a solvability condition is satisfied. The solvability condition demands that the secular term in the right hand of the equation (3.22) be orthogonal to every solution of the homogeneous problem. In order to avoid the unbounded solution, the solvability condition is derived as follows:

$$-\frac{dA_k}{dT_1} (2i\omega_k \langle I\phi_k, \phi_k \rangle + \langle M\phi_k, \phi_k \rangle) - \delta c \tilde{A}_k e^{i\mu T_1} \left\langle \left( \frac{\Omega}{2} - \omega_k \right) \phi_k' - i c \phi_k'' \right\rangle$$

$$+ \frac{1}{2} \left[ \frac{(2i\omega_k + a\alpha)}{4i\omega_k + 2\alpha} + a \right] A_k^2 \tilde{A}_k \langle 2\phi_k' \phi_k'' + \phi_k'^2 \phi_k''' \rangle = 0.$$ \hfill (3.23)

Based on the natural frequencies and the eigenfunctions given by equation (3.16), evaluating the inner products in equation (3.23) gives

$$\langle 2\phi_k' \phi_k'' + \phi_k'^2 \phi_k''' \rangle = \frac{1}{2} \pi^4 k^4 (3 + 2c^2 + 3c^4),$$

$$\langle \phi_k', \phi_k \rangle = 0, \quad \langle \phi_k'', \phi_k \rangle = \frac{\pi k}{2c} (e^{-2ic\pi} - 1).$$ \hfill (3.24)

Substitution of equations (3.16) and (3.24) into equation (3.23) results in

$$\frac{dA_k}{dT_1} + \kappa_1 \tilde{A}_k e^{i\mu T_1} + \kappa_2 A_k^2 \tilde{A}_k = 0,$$  \hfill (3.25)

where

$$\kappa_1 = \frac{i}{4} c \delta (e^{-2ic\pi} - 1),$$  \hfill (3.26)

$$\kappa_2 = -\frac{i}{8} \left[ a + \frac{2ik(1-c^2) + a\alpha}{4ik(1-c^2) + 2a} \right] k^3 \pi^3 (3 + 2c^2 + 3c^4).$$ \hfill (3.27)
The non-trivial solution $A_k$ is expressed in the polar form

$$A_k = \alpha_k e^{i\beta_k}.$$  

Then, $\alpha_k$ and $\beta_k$ represent the amplitude and the phase angle of the response, respectively. Substituting equation (3.28) into equation (3.25), separating the resulting equation into real and imaginary parts and solving the derivatives of $\alpha_k$ and $\beta_k$ with respect to $T_1$ from the resulting equations, leads to

$$\frac{d\alpha_k}{dT_1} = \alpha_k [\text{Im}(\kappa_1) \sin \theta_k - \text{Re}(\kappa_1) \cos \theta_k] - \text{Re}(\kappa_2) \alpha_k^3,$$

$$\frac{d\beta_k}{dT_1} = \mu + 2[\text{Re}(\kappa_1) \sin \theta_k + \text{Im}(\kappa_1) \cos \theta_k] + 2 \text{Im}(\kappa_2) \alpha_k^2,$$

where

$$\theta_k = \mu T_1 - 2\beta_k.$$  

The real and imaginary parts of $\kappa_1$ and $\kappa_2$ can be calculated from equations (3.26) and (3.27):

$$\text{Re}(\kappa_1) = \frac{1}{4} \delta c \sin(2ck\pi), \quad \text{Im}(\kappa_1) = \frac{1}{4} \delta c [\cos(2ck\pi) - 1],$$  

$$\text{Re}(\kappa_2) = \frac{1}{8} \frac{\alpha(1-a)(1-c^2)}{4k^2 \pi^2 (1-c^2)^2 + \alpha^2} k^4 \pi^4 (3 + 2c^2 + 3c^4),$$

$$\text{Im}(\kappa_2) = -\frac{1}{8} \left[ \left( a + \frac{4k^2 \pi^2 (1-c^2)^2 + a^2 \alpha^2}{8k^2 \pi^2 (1-c^2)^2 + 2\alpha^2} \right)^3 \pi^3 (3 + 2c^2 + 3c^4) \right].$$  

Note that $\text{Re}(\kappa_2)$ is always positive and $\text{Im}(\kappa_2)$ is always negative.

The Kirchhoff model, equation (2.11), can be analysed using the same procedure of the method of multiple scales. A similar solvability condition can be derived. In fact, the amplitude and the phase angle of its non-trivial response in principal parametric resonance are also described by equation (3.25), while coefficient $\kappa_2$ is defined by

$$\kappa_2 = -\frac{i}{4} \left\{ a k^3 \pi^3 (1 + c^2)^2 + \frac{2ik\pi(1 - c^2) + a\alpha}{4c^2[2ik\pi(1 - c^2) + \alpha]} \right\} k\pi(1 - \cos 2ck\pi),$$  

instead of equation (3.27). The real and imaginary parts of $\kappa_2$ are, respectively,

$$\text{Re}(\kappa_2) = \frac{\alpha(1-a)(1-c^2)}{8c^2[4k^2 \pi^2 (1-c^2)^2 + \alpha^2]} k\pi(1 - \cos 2ck\pi),$$

$$\text{Im}(\kappa_2) = -\frac{1}{4} \left\{ a k^3 \pi^3 (1 + c^2)^2 + \frac{4k^2 \pi^2 (1-c^2)^2 + a\alpha^2}{4c^2[4k^2 \pi^2 (1-c^2)^2 + \alpha^2]} \right\} k\pi(1 - \cos 2ck\pi).$$  

Similar to those for the Mote model, $\text{Re}(\kappa_2)$ is always positive and $\text{Im}(\kappa_2)$ is always negative.

4. Steady-state periodical responses and their stability

Equation (3.25) possesses a trivial equilibrium point at the origin, which represents the string at the straight equilibrium configuration undergoing no transverse vibration. However, the trivial equilibrium is not always stable. Linearization of equation (3.25) at the origin leads to

$$\frac{dA_k}{dT_1} + \kappa_1 \dot{A}_k e^{i\mu T_1} = 0.$$  \hspace{1cm} (4.1)

Suppose that the perturbed solutions of equation (4.1) take the form

$$A_k = (a_r + ia_i)e^{i\mu T_1/2},$$  \hspace{1cm} (4.2)

where $a_r$ and $a_i$ are real functions of $T_1$. Substitution of equation (4.2) into equation (4.1), and separation of the real and imaginary parts in the resulting equations lead to

$$\begin{align*}
\frac{da_r}{dT_1} &= -\text{Re}(\kappa_1)a_r + \left[\frac{\mu}{2} - \text{Im}(\kappa_1)\right]a_i, \\frac{da_i}{dT_1} &= -\left[\frac{\mu}{2} + \text{Im}(\kappa_1)\right]a_r + \text{Re}(\kappa_1)a_i.
\end{align*}$$  \hspace{1cm} (4.3)

The stability of the zero solution to equation (4.3) is determined by the eigenvalues of its coefficient matrix. The eigenvalues $\lambda_{1,2}$ can be calculated as

$$\lambda_{1,2}^2 = |\kappa_1|^2 - \left(\frac{\mu}{2}\right)^2.$$  \hspace{1cm} (4.4)

Therefore, the trivial zero solution to equation (4.3) is unstable if

$$-2|\kappa_1| < \mu < 2|\kappa_1|.$$  \hspace{1cm} (4.5)

According to the Lyapunov linearized stability theory, the original nonlinear system, equation (3.25), possesses an unstable zero solution. Physically, condition (4.5) means that the equilibrium of the string is unstable in principal parametric resonance, if the frequency of the axial-speed fluctuation is close enough to two times one of natural frequencies of the corresponding linear system.

The non-trivial equilibrium points of equation (3.29), at which the amplitude $\alpha_k$ and the new phase angle $\theta_\kappa$ are constant, correspond to the steady-state periodic transverse vibration of the string. Let $\alpha_k$ and $\theta_\kappa$ be constant, and equation (3.29) then yields

$$\begin{align*}
\alpha_k[&\text{Im}(\kappa_1)\sin \theta_\kappa - \text{Re}(\kappa_1)\cos \theta_\kappa] - \text{Re}(\kappa_2)\alpha_k^2 = 0, \\mu &+ 2[\text{Re}(\kappa_1)\sin \theta_\kappa + \text{Im}(\kappa_1)\cos \theta_\kappa] + 2i\text{Im}(\kappa_2)\alpha_k^2 = 0.
\end{align*}$$  \hspace{1cm} (4.6)

Eliminating $\theta_\kappa$ from equation (4.6) gives

$$\left[\text{Im}(\kappa_2)\alpha_k^2 + \frac{\mu}{2}\right]^2 + \text{Re}(\kappa_2)\alpha_k^2 = |\kappa_1|^2.$$  \hspace{1cm} (4.7)
The amplitudes of the steady-state periodic transverse vibration can be solved from equation (4.7) as

$$\alpha_{k1,2} = \frac{-\mu \text{Im}(k_2) \pm \sqrt{4|\kappa_1|^2|\kappa_2|^2 - \mu^2 \text{Re}^2(k_2)}}{\sqrt{2|\kappa_2|}}. \quad (4.8)$$

From equation (4.8), the non-trivial solutions exist on condition that

$$-\mu \text{Im}(k_2) \pm \sqrt{4|\kappa_1|^2|\kappa_2|^2 - \mu^2 \text{Re}^2(k_2)} > 0 \quad (4.9)$$

and

$$4|\kappa_1|^2|\kappa_2|^2 - \mu^2 \text{Re}^2(k_2) \geq 0. \quad (4.10)$$

Using the facts that \(\text{Re}(k_2) > 0\) and \(\text{Im}(k_2) < 0\), one can derive the existence condition of the first (larger amplitude) periodical response from inequalities (4.9) and (4.10) as follows:

$$-2|\kappa_1| \leq \mu \leq \frac{2|\kappa_1||\kappa_2|}{\text{Re}(k_2)}, \quad (4.11)$$

and the existence condition of the second (smaller amplitude) periodical response as

$$2|\kappa_1| \leq \mu \leq \frac{2|\kappa_1||\kappa_2|}{\text{Re}(k_2)}. \quad (4.12)$$

Inequalities (4.11) and (4.12) show that the first and the second non-trivial solutions have an identical upper boundary of existence conditions that depends on the viscoelastic parameters \(a\) and \(\alpha\), while the lower boundaries of existence conditions for the first and the second non-trivial solutions have no relation with the viscoelastic parameters. In addition, inequalities (4.5), (4.11) and (4.12) indicate that the lower boundaries of existence conditions for the first and the second non-trivial solutions are the same as the lower and upper boundaries of instability conditions for the trivial solution, respectively.

From the Lyapunov linearized stability theory, the stability of the non-trivial equilibriums can be determined by the nature of the eigenvalues of the Jacobian \(J\) calculated at the fixed points \((\alpha_{k1,2}, \theta_{k1,2})\),

$$J = \begin{pmatrix} \text{Im}(\kappa_1) \sin \theta_{k1,2} - \text{Re}(\kappa_1) \cos \theta_{k1,2} & \alpha_{k1,2} \text{Im}(\kappa_1) \cos \theta_{k1,2} + \text{Re}(\kappa_1) \sin \theta_{k1,2} \\ -3 \text{Re}(\kappa_2) \alpha_{k1,2}^2 & 4 \text{Im}(\kappa_2) \alpha_{k1,2} \\ 4 \text{Im}(\kappa_2) \alpha_{k1,2} & 2[\text{Re}(\kappa_1) \cos \theta_{k1,2} - \text{Im}(\kappa_1) \sin \theta_{k1,2}] \end{pmatrix}, \quad (4.13)$$

where \((\alpha_{k1,2}, \theta_{k1,2})\) is satisfied with equation (4.6). If both eigenvalues of the matrix (4.13) have negative real parts, then the fixed point is stable. Thus, the corresponding periodical transverse vibration is stable. If there is an eigenvalue with a positive real part, then the fixed point is unstable, as is the corresponding

periodical transverse vibration. In fact, the characteristic equation is
\[ \lambda^2 + 4 \Re(\kappa_2)\alpha_{k1,2}^2 \lambda + 4|\kappa_2|^2\alpha_{k1,2}^4 + 2 \Im(\kappa_2)\alpha_{k1,2}^2 \mu = 0. \]  
(4.14)
Because \( \mu \) is given by equation (4.7), the detuning parameter \( \mu \) should be
\[ \mu = 2 \left\{ -\Im(\kappa_2)\alpha_{k1,2}^2 \pm \sqrt{|\kappa_1|^2 - \Re^2(\kappa_2)\alpha_{k1,2}^4} \right\}. \]
(4.15)
Substitution of equation (4.15) into equation (4.14) gives
\[ \lambda^2 + 4\Re(\kappa_2)\alpha_{k1,2}^2 \lambda + 4\Re^2(\kappa_2)\alpha_{k1,2}^4 \]
\[ \mp 4\Im(\kappa_2)\alpha_{k1,2}^2 \sqrt{|\kappa_1|^2 - \Re^2(\kappa_2)\alpha_{k1,2}^4} = 0. \]  
(4.16)
For the first non-trivial solution, which is
\[ 4\Re(\kappa_2)\alpha_{k1}^2 > 0, \quad 4\Re^2(\kappa_2)\alpha_{k1}^4 - 4\Im(\kappa_2)\alpha_{k1}^2 \sqrt{|\kappa_1|^2 - \Re^2(\kappa_2)\alpha_{k1}^4} > 0, \]  
(4.17)
both eigenvalues have negative real parts. For the second non-trivial solution, it can be proved that
\[ 4\Re^2(\kappa_2)\alpha_{k2}^4 + 4\Im(\kappa_2)\alpha_{k2}^2 \sqrt{|\kappa_1|^2 - \Re^2(\kappa_2)\alpha_{k2}^4} < 0. \]  
(4.18)
In fact, let
\[ f(\alpha_{k2}^2) = 4\Re^2(\kappa_2)\alpha_{k2}^4 + 4\Im(\kappa_2)\alpha_{k2}^2 \sqrt{|\kappa_1|^2 - \Re^2(\kappa_2)\alpha_{k2}^4}. \]  
(4.19)
If
\[ \mu = \frac{2|\kappa_1||\kappa_2|}{\Re(\kappa_2)}, \]  
(4.20)
then \( \alpha_{k2}^2 \) achieves the maximum
\[ \alpha_{k2,m}^2 = -\frac{|\kappa_1|\Im(\kappa_2)}{\Re(\kappa_2)|\kappa_2|}. \]  
(4.21)
It is easy to check that
\[ f(\alpha_{k2,m}^2) = 0. \]  
(4.22)
In addition, \( f \) is a monotonically increasing function on \((0, -|\kappa_1|\Im(\kappa_2)/\Re(\kappa_2)|\kappa_2|)\), because its derivative with respect to \( \alpha_{k2}^2 \) is
\[ \frac{df}{d\alpha_{k2}^2} = 8\Re^2(\kappa_2)\alpha_{k2}^2 + 4\Im(\kappa_2)\alpha_{k2}^4 \left| \kappa_1 \right|^2 - 2\Re^2(\kappa_2)\alpha_{k2}^4 \sqrt{\left| \kappa_1 \right|^2 - \Re^2(\kappa_2)\alpha_{k2}^4} > 0. \]  
(4.23)
Thus, inequality (4.18) holds. Therefore, according to the Routh–Hurwitz criterion, the first non-trivial equilibrium is always stable and the second non-trivial equilibrium is always unstable.

The preceding analysis indicates that the steady-state periodic transverse vibration of the string can possibly occur if the frequency of the axial-speed
fluctuation is near enough to two times one of the natural frequencies of the corresponding linear system, while the trivial equilibrium loses its stability. The dependence of the amplitude on the detuning parameter is shown in figure 2. In figure 2, the two first-order principal parametric resonance are investigated based on Mote’s model, equation (3.2), in which all other parameters are fixed as $c=0.2$, $\delta=0.5$, $\alpha=5$, $a=0.1$. Solid lines represent the first non-trivial solutions (stable), and dash-dot lines represent the second non-trivial (unstable) solutions. The cross points, actually the connection points, of the two solutions are given by equations (4.20) and (4.21). Based on equation (4.7), it can be calculated that

$$\frac{d\alpha_k}{d\mu} = \frac{-\mu}{4\alpha_k^2|\kappa^2|\alpha_k^2 + \mu \text{Im} (\kappa_2)}.$$  \hfill (4.24)

Hence, the derivative tends to infinite at the connection points. That is, the tangent lines at the connection points are parallel to axis $\alpha_k$.

5. Numerical results of the two models

In this section, numerical results of steady-state responses, their existence boundaries and their stability will be presented for the two first-order principal parametric resonances of the axially moving string. Effects of the mean axial speed, the axial-speed fluctuation amplitude and the viscoelastic parameters on the periodical response are investigated based on Mote’s model and Kirchhoff’s model, respectively. In all figures, the solid lines and the dash-dot lines represent stable and unstable non-trivial responses.

In the computation using Mote’s model, the fixed parameters are selected as $c=0.2$, $\delta=1.0$, $\alpha=5$ and $a=0.1$, if no other values are assigned. Figure 3 illustrates the effects of the mean axial speed $c$, where three different values of $c$ are chosen as 0.1, 0.2 and 0.3. Both the amplitudes and the existence intervals of periodical responses increase with the growth of the mean axial-speed fluctuation. Figure 4 demonstrates the steady-state amplitude varying with the detuning parameter for three different values of the axial-speed fluctuation.

Figure 2. The steady-state amplitude varying with the detuning parameter: (a) $k=1$; (b) $k=2$. 

![Figure 2](image-url)
amplitude $\delta$—namely, 0.4, 1.0 and 1.6. The larger axial-speed fluctuation amplitude leads to the larger amplitudes and the existence intervals of periodical responses. Figure 5 depicts the effects of the viscoelastic parameter $\alpha$, where three different values of $\alpha$ are chosen as 0.001, 0.1 and 0.3. With the increase of the viscoelastic parameter $\alpha$, the periodical responses amplitudes decrease, whereas the upper boundaries of their existence conditions augment. Figure 6 shows the effects of the parameter $\alpha$, where three different values of $\alpha$ are chosen as 2.0, 3.0 and 5.0 for the first parametric resonance and 2.0, 5.0 and 10.0 for the second parametric resonance. With the increase of the viscoelastic parameter $\alpha$, the upper boundaries of their existence conditions diminish, whereas the periodical response amplitudes change slightly. In all cases, the periodical response amplitudes of the first parametric resonance are much larger than those of the second parametric resonance, whereas the existence intervals of the first parametric resonance are smaller than those of the second parametric resonance.
Figure 5. The effects of the viscoelastic parameter $a$ (Mote's model): (a) $k=1$; (b) $k=2$.

Figure 6. The effects of the viscoelastic parameter $\alpha$ (Mote's model): (a) $k=1$; (b) $k=2$.

Figure 7. The effects of the mean axial speed $c$ (Kirchhoff's model): (a) $k=1$; (b) $k=2$. 
Figure 8. The effects of the axial-speed variation amplitude $\delta$ (Kirchhoff's model): (a) $k=1$; (b) $k=2$.

Figure 9. The effects of the viscoelastic parameter $a$ (Kirchhoff's model): (a) $k=1$; (b) $k=2$.

Figure 10. The effects of the viscoelastic parameter $\alpha$ (Kirchhoff's model): (a) $k=1$; (b) $k=2$. 
In the computation using Kirchhoff’s model, the fixed parameters are selected as $c = 0.15$, $\delta = 1.0$, $\alpha = 5$ and $\alpha = 0.1$, if no other values are assigned. Figures 7–10 present the effects of the mean axial speed $c$, the axial-speed fluctuation amplitude $\delta$ and the viscoelastic parameters $\alpha$ and $\alpha$, respectively. Numerical calculations show that the Mote and Kirchhoff’s models yield the qualitatively same results. In other words, the two models produce similar results for the amplitudes and the existence boundaries of steady-state periodical responses that change with parameters. However, the two models are quantitatively different. For given parameters $c = 0.2$, $\delta = 1.0$, $\alpha = 5$ and $\alpha = 0.1$, the results based on the two models are compared in figure 11. The Kirchhoff model predicts the larger periodical response amplitudes, while the existence intervals yielded by the two models are almost the same.

6. Conclusions

This paper explores nonlinear parametric vibration of axially accelerating strings constituted by the Boltzmann superposition principle. The Mote model governing transverse vibration is derived from the Eulerian equation of motion of a continuum, the constitutive relationship and the strain–displacement relationship. Under the assumption that the string tension can be replaced by the averaged tension over the string, the equation reduces the Kirchhoff model. The two models are analysed via the method of multiple scales in principal parametric resonance. The non-trivial steady-state responses and their existence conditions are presented. The Lyapunov linearized stability theory is applied to obtain the stability conditions of straight equilibrium and non-trivial steady-state response. The investigation demonstrates that an instability interval of the detuning parameters exists on which the straight equilibrium is unstable, and the first (second) non-trivial steady-state response is always stable (unstable). Numerical calculations yield the following conclusions.

(i) Qualitatively, the two models predict the same changing tendencies of the amplitudes and the existence intervals of steady-state periodical responses with parameters.
(ii) Quantitatively, the Kirchhoff model predicts larger periodical response amplitudes, whereas the existence intervals determined by the two models are essentially the same.

(iii) The periodical response amplitudes increase, with the growth of the mean axial speed and the axial-speed fluctuation amplitude and with the decrease of viscoelastic parameters $a$.

(iv) The lower boundaries of the existence intervals for periodical responses decrease with the growth of the mean axial speed and the axial-speed fluctuation amplitude.

(v) The upper boundaries of the existence intervals for periodical responses increase, with the growth of the mean axial speed, the axial-speed fluctuation amplitude, the viscoelastic parameter $a$ and the decrease of the viscoelastic parameter $\alpha$.

(vi) The periodical response amplitudes of the first parametric resonance are much larger than those of the second parametric resonance, whereas the existence intervals of the first parametric resonance are smaller than those of the second parametric resonance.

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References


